

FULL EXTREMAL PROCESS, CLUSTER LAW AND FREEZING FOR TWO-DIMENSIONAL DISCRETE GAUSSIAN FREE FIELD

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Abstract: We study the extremal process associated with the Discrete Gaussian Free Field (DGFF) in scaled-up (square-)lattice versions of bounded open planar domains subject to mild regularity conditions on the boundary. We prove that, in the scaling limit, this process tends to a Cox process decorated by independent, correlated clusters whose distribution is completely characterized. As an application, we control the scaling limit of the discrete supercritical Liouville measure, extract a Poisson-Dirichlet statistics for the limit of the Gibbs measure associated with the DGFF and establish the “freezing phenomenon” conjectured to occur in the “glassy” phase. In addition, we prove a local limit theorem for the position and value of the absolute maximum. The proofs are based on a concentric, finite-range decomposition of the DGFF and entropic-repulsion arguments for an associated random walk. Although we naturally build on our earlier work on this problem, the methods developed here are largely independent.

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1. INTRODUCTION

Recent years have witnessed remarkable advances in the understanding of extreme values of the two-dimensional Discrete Gaussian Free Field (DGFF). This is a Gaussian process $\{h_x : x \in V\}$ in a proper subset V of the square lattice \mathbb{Z}^2 such that

$$E(h_x) = 0 \quad \text{and} \quad E(h_x h_y) = G^V(x, y), \quad (1.1)$$

where G^V denotes the Green function of the simple symmetric random walk in V killed upon exit from V . (We think of h_x as fixed to zero outside V .) Early efforts focused on the absolute maximum in square domains $V_N := (0, N)^2 \cap \mathbb{Z}^2$. Writing $g := 2/\pi$ for the constant in the asymptotic $G^{V_N}(x, x) = g \log N + O(1)$ whenever N is large and x is deep inside V_N , and denoting

$$m_N := 2\sqrt{g} \log N - \frac{3}{4}\sqrt{g} \log \log N, \quad (1.2)$$

from the works of Bolthausen, Deuschel and Zeitouni [12], Bramson and Zeitouni [16] and, particularly, Bramson, Ding and Zeitouni [15] we now know that the law of $\max_{x \in V_N} h_x - m_N$ converges to a non-degenerate limit as $N \rightarrow \infty$.

In [10, 11] the present authors turned to the extremal process associated with the DGFF in a sequence $\{D_N\}$ of scaled-up versions of a bounded open set $D \subset \mathbb{C}$ (see (2.1–2.2) for precise definitions). Writing δ_a for the Dirac point-mass at a , a standard way to describe extreme order statistics is to encode both the scaled positions and the centered values of the field $\{h_x : x \in D_N\}$ into the random point measure

$$\eta_N^D := \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_x - m_N} \quad (1.3)$$

on $D \times \mathbb{R}$ and study its distributional limits as $N \rightarrow \infty$. However, for conceptual reasons as well as the technical nature of the approach, the analysis [10, 11] addressed only *local* maxima (i.e., the tips of large “peaks”) of the field. Using

$$\Lambda_r(x) := \{z \in \mathbb{Z}^2 : |z - x| \leq r\}, \quad (1.4)$$

to describe the meaning of the word “local,” these were captured by the random point measure $\tilde{\eta}_{N,r}^D$ on $D \times \mathbb{R}$ defined by

$$\tilde{\eta}_{N,r}^D := \sum_{x \in D_N} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N}. \quad (1.5)$$

For any sequence $r_N \rightarrow \infty$ with $N/r_N \rightarrow \infty$, it was then shown that

$$\tilde{\eta}_{N,r_N}^D \xrightarrow[N \rightarrow \infty]{\text{law}} \text{PPP}(Z^D(dx) \otimes e^{-\alpha h} dh), \quad (1.6)$$

where $\text{PPP}(\lambda)$ denotes the Poisson point process with intensity measure λ ,

$$\alpha := \frac{2}{\sqrt{g}} = \sqrt{2\pi} \quad (1.7)$$

and Z^D is a *random* Borel measure on D with full support and $0 < Z^D(D) < \infty$ a.s. This measure is independent of the sequence r_N .

The laws of the measures Z^D obey a host of specific properties that characterize them uniquely up to an overall multiplicative constant (see Theorem 2.8 of [11]). Despite its restriction to local maxima, (1.6) yields interesting conclusions for the full process η_N^D as well, e.g., the limit distribution of the scaled position and centered value of the absolute maximum,

$$P\left(N^{-1} \operatorname{argmax}_{D_N} h \in A, \max_{x \in D_N} h(x) - m_N \leq t\right) \xrightarrow[N \rightarrow \infty]{} E\left(\widehat{Z}^D(A) e^{-\alpha^{-1} e^{-\alpha t} Z^D(D)}\right) \quad (1.8)$$

for $A \subset D$ open and any $t \in \mathbb{R}$, as well as joint laws of maxima in any finite number of disjoint subsets of D_N . Unfortunately, the methods of [10, 11], being tailored to the global structure of the extremal points, do not generalize to include local information.

The aim of the present article is to complete the description started in [10, 11] and derive the distributional limit of the *full* extremal process (1.3). This requires development of techniques that capture the local structure of the extreme points and are, for reasons just mentioned, thus more or less unrelated to those of [10, 11]. But then, as a reward, we are able to establish a number of additional results that have been conjectured in the so called “glassy” phase for the Gibbs measure naturally associated with the DGFF. Our approach also yields a *local* limit theorem for the location and the value of the absolute maximum.

2. MAIN RESULTS

We proceed to we give precise statements of our results. The structure of the proofs, which constitute the remainder of this paper, is outlined along with some heuristics in Section 2.4.

2.1 Full extremal process.

The description of our results naturally starts with a limit theorem for the full extremal process. We will follow the setting of Biskup and Louidor [11] that considers the DGFF over scaled-up versions of rather general domains in the complex plane.

Let \mathfrak{D} be the class of all bounded open sets $D \subset \mathbb{C}$ with a finite number of connected components and with boundary ∂D that has only a finite number of connected components with each having a positive (Euclidean) diameter. Given $D \in \mathfrak{D}$, let $\{D_N\}$ be a sequence such that

$$D_N \subseteq \{x \in \mathbb{Z}^2 : d(x/N, \Delta^c) > 1/N\} \quad (2.1)$$

and, for each $\delta > 0$ and all N sufficiently large, also

$$D_N \supseteq \{x \in \mathbb{Z}^2 : d(x/N, \Delta^c) > \delta\}. \quad (2.2)$$

Note that $x \in D_N$ implies $x/N \in D$. A key point is that (1.6) holds for every $D \in \mathfrak{D}$ (cf Theorem 2.1 of Biskup and Louidor [11] for a formal statement).

It is clear that the values of the field at nearby vertices are heavily correlated. Each high value of the field will thus come with a whole cluster of comparable values at basically the same spatial location. For this reason, instead of (1.3), it is more natural to capture the extremal process associated with the DGFF in D_N by way of *structured* extremal point measures given by

$$\eta_{N,r}^D := \sum_{x \in D_N} 1_{\{h_x = \max_{z \in \Lambda_r(x)} h_z\}} \delta_{x/N} \otimes \delta_{h_x - m_N} \otimes \delta_{\{h_x - h_{x+z} : z \in \mathbb{Z}^2\}}. \quad (2.3)$$

These are formally Radon measures on $D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$ (with the product topology on $\mathbb{R}^{\mathbb{Z}^2}$) that extend the point measures from (1.5) by including control of the “shape” of the field “around” the local maxima.

The space of Radon measures on $D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$ is naturally endowed with the topology of vague convergence which, in turn, permits discussion of distributional limits. (A sequence of random Radon measures thus converges in distribution if, and only if, integrals of compactly-supported continuous functions converge in distribution.) Our principal result is then:

Theorem 2.1 (Full scaling limit) *For each $D \in \mathfrak{D}$, let Z^D be the random Borel measure on D for which (1.6) holds. There is a probability measure ν on $[0, \infty)^{\mathbb{Z}^2}$ such that for each $D \in \mathfrak{D}$ and each r_N with $r_N \rightarrow \infty$ and $r_N/N \rightarrow 0$,*

$$\eta_{N,r_N}^D \xrightarrow[N \rightarrow \infty]{\text{law}} \text{PPP}(Z^D(dx) \otimes e^{-\alpha h} dh \otimes \nu(d\phi)), \quad (2.4)$$

where α is as in (1.7). Moreover, $\phi_0 = 0$ and $\{x \in \mathbb{Z}^2 : \phi_x \leq c\} < \infty$ ν -a.s. for each $c > 0$.

To interpret this result, one can say that although the spatial positions of the local maxima are correlated via the random measure Z^D , the configurations around each of the local maxima — the shapes of the nearly-highest peaks — are (in the limit) independent samples from ν .

A consequence of the above theorem is a representation of the limit law of the “unstructured” extremal process η_N^D from (1.3) by means of a *cluster process*:

Corollary 2.2 (Cluster process) *For the setting and notation of Theorem 2.1, let $\{(x_i, h_i) : i \in \mathbb{N}\}$ enumerate the points in a sample from $\text{PPP}(Z^D(dx) \otimes e^{-\alpha h} dh)$. Let $\{\phi_z^{(i)} : z \in \mathbb{Z}^2\}$, $i \in \mathbb{N}$, be independent samples from the measure ν , independent of $\{(x_i, h_i) : i \in \mathbb{N}\}$. Then*

$$\eta_N^D \xrightarrow[N \rightarrow \infty]{\text{law}} \sum_{i \in \mathbb{N}} \sum_{z \in \mathbb{Z}^2} \delta_{(x_i, h_i - \phi_z^{(i)})}. \quad (2.5)$$

The measure on the right is locally finite on $D \times \mathbb{R}$ a.s.

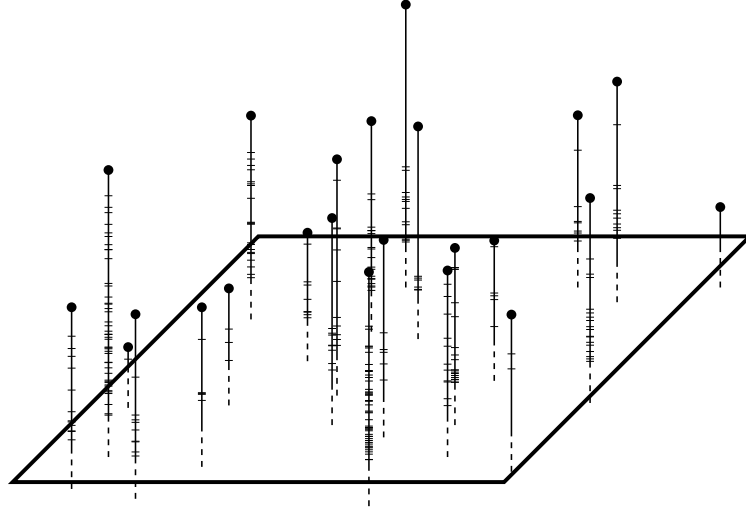


FIG. 1: An illustration of the limit (2.5) of the unstructured point process η_N^D for D being a unit square. The “notches” on the vertical lines mark the locations of sample points of an individual cluster. Only the points above a certain (arbitrary) base value are included. The top values in the clusters (marked by bullets) are distributed according to the Cox process in (1.6).

When we disregard the spatial positions, (2.5) becomes even simpler:

$$\sum_{x \in D_N} \delta_{h_x - m_N} \xrightarrow[N \rightarrow \infty]{\text{law}} \sum_{i \in \mathbb{N}} \sum_{z \in \mathbb{Z}^2} \delta_{t_i + \alpha^{-1} \log Z^D(D) - \phi_z^{(i)}} \quad (2.6)$$

where $\{t_i : i \in \mathbb{N}\}$ is a sample from Gumbel PPP($e^{-\alpha h} dh$), $\{\phi_z^{(i)} : z \in \mathbb{Z}^2\}$ are i.i.d. samples from ν and $Z^D(D)$ is the total mass of $Z^D(dx)$, with all three objects independent of one another. The limit process is thus a randomly-shifted Gumbel process decorated by independent and identically distributed clusters.

A randomly-shifted, i.i.d.-decorated Gumbel process is the limit of the extremal process associated with the Branching Brownian motion; see Arguin, Bovier and Kistler [6–8], Aïdekon, Berestycki, Brunet and Shi [5] or Bovier and Hartung [13] who even track additional information analogous to our spatial positions. In these studies the cluster law is defined by taking the whole ensemble of branching Brownian motions conditioned to have an excessively large absolute maximum. (This essentially forces the whole process to be just one cluster.) It turns out that a relatively explicit description of the cluster law is possible in our case as well.

Let ν^0 be the law of the mean-zero DGFF in \mathbb{Z}^2 pinned to zero at $x = 0$ or, equivalently, the DGFF in $\mathbb{Z}^2 \setminus \{0\}$. (Recall that all of our DGFFs have zero boundary conditions.) Explicitly, ν^0 is a Gaussian law on $\mathbb{R}^{\mathbb{Z}^2}$ with mean zero and covariance

$$\text{Cov}_{\nu^0}(\phi_x, \phi_y) = \mathfrak{a}(x) + \mathfrak{a}(y) - \mathfrak{a}(x - y), \quad (2.7)$$

where $\mathfrak{a}: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is the potential kernel of the simple symmetric random walk started from zero with the explicit representation

$$\mathfrak{a}(x) := \int_{[-\pi, \pi]^2} \frac{dk}{(2\pi)^2} \frac{1 - \cos(k \cdot x)}{\sin(k_1/2)^2 + \sin(k_2/2)^2}. \quad (2.8)$$

Note that $\phi_0 = 0$ ν^0 -a.s. Then we have:

Theorem 2.3 (Cluster law) *The measure ν in Theorem 2.1 is given by the weak limit*

$$\nu(\cdot) = \lim_{r \rightarrow \infty} \nu^0 \left(\phi + \frac{2}{\sqrt{g}} \mathfrak{a} \in \cdot \mid \phi_x + \frac{2}{\sqrt{g}} \mathfrak{a}(x) \geq 0: |x| \leq r \right). \quad (2.9)$$

In particular, the limit exists and is non-degenerate.

The strong FKG property associated with the DGFF (cf Lemma B.8) shows that, for increasing events, the probability on the right of (2.9) is non-decreasing in r . Unfortunately, this is not enough to infer the existence of the limit as stated: As the space of configurations is not compact, work is needed to prevent blow-ups to infinity (i.e., to prove tightness on $[0, \infty)^{\mathbb{Z}^2}$). To see that this is in fact a subtle issue, we note:

Theorem 2.4 *There is a constant $\tilde{c}_* \in (0, \infty)$ such that*

$$\nu^0 \left(\phi_x + \frac{2}{\sqrt{g}} \mathfrak{a}(x) \geq 0: |x| \leq r \right) \sim \frac{\tilde{c}_*}{(\log r)^{1/2}}, \quad r \rightarrow \infty. \quad (2.10)$$

Thus, staying non-negative in larger and larger volumes is increasingly costly for $\phi + \frac{2}{\sqrt{g}} \mathfrak{a}$ in spite of the “logarithmic boost” received from \mathfrak{a} (recall that $\mathfrak{a}(x) = g \log |x| + O(1)$ as $|x| \rightarrow \infty$). Consequently, one cannot pass the limit $r \rightarrow \infty$ inside the conditioning event in (2.9) and the measures ν and ν^0 are supported on disjoint sets.

We remark that, in the proof of Theorem 2.4, the constant \tilde{c}^* is obtained as the $\ell \rightarrow \infty$ limit of the quantity $\Xi_\ell^{\text{in}}(f)$ from (5.1) for $f := 1$. However, we do not seem to have a way to express this constant without a limit procedure.

2.2 Local limit theorem for absolute maximum.

The proofs of the above theorems hinge on control of the DGFF conditioned on the maximum occurring at a given point. This is achieved by way of (what we call) a concentric decomposition of the DGFF; see Section 3. An augmented version of the same argument then yields also a *local* limit theorem for both the value and the position of the absolute maximum:

Theorem 2.5 (Local limit law for absolute maximum) *For each $D \in \mathfrak{D}$ there exists a continuous function $\rho^D: D \times \mathbb{R} \rightarrow [0, \infty)$ such that for each $\{D_N\}$ that obeys (2.1–2.2), each $a < b$ and uniformly in x over compact subsets of D ,*

$$\lim_{N \rightarrow \infty} N^2 P \left(\arg\max_{D_N} h = \lfloor xN \rfloor, \max_{D_N} h - m_N \in (a, b) \right) = \int_a^b \rho^D(x, t) dt. \quad (2.11)$$

Moreover, $x \mapsto \rho^D(x, t)$ is for each $t \in \mathbb{R}$ the Radon-Nikodym derivative of the measure

$$A \mapsto e^{-\alpha t} E \left(Z^D(A) e^{-\alpha^{-1} e^{-\alpha t} Z^D(D)} \right) \quad (2.12)$$

with respect to the Lebesgue measure on \mathbb{R}^2 . Here Z^D is the random measure from (1.6).

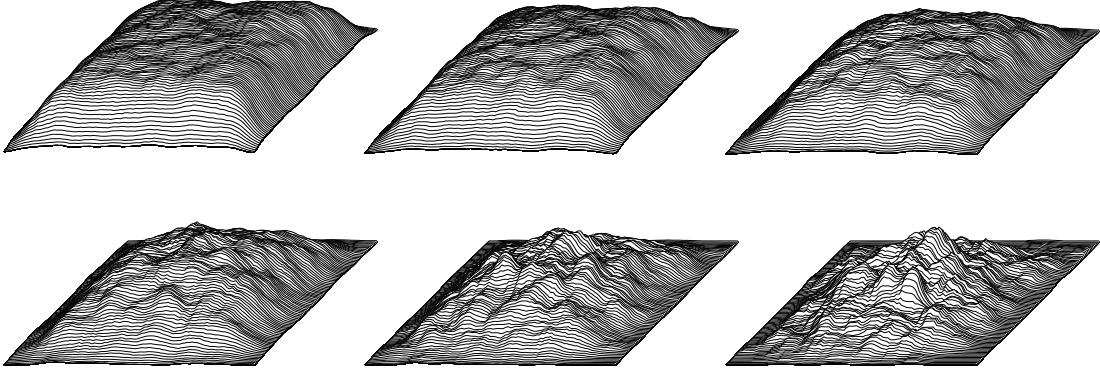


FIG. 2: Empirical plots of $x \mapsto \rho^D(x, t)$ obtained from a set of about 100000 samples of the maximum of the DGFF on a 100×100 square. The plots (labeled left to right starting with the top row) correspond to t increasing by uniform amounts over an interval of length 3 with t in the fourth figure set to the empirical mean. A certain amount of smoothing has been applied to eliminate discrete effects.

Comparing (2.11–2.12) with (1.8), the local limit theorem is consistent with the limit law of the scaled and centered maximum. The large- t asymptotic of $\rho^D(x, t)$ is thus known; indeed, from Biskup and Louidor [11, Theorem 2.6] we infer that

$$\rho^D(x, t) \sim t e^{-\alpha t} \psi^D(x), \quad t \rightarrow \infty, \quad (2.13)$$

where, for D simply connected,

$$\psi^D(x) = c_* \text{rad}_D(x)^2 \quad (2.14)$$

with c_* a positive constant and $\text{rad}_D(x)$ denoting the conformal radius of D from x . Our proof gives a formula for $\rho^D(x, t)$ (see (6.31)) but this is still quite inexplicit as singular limits remain involved. (Notwithstanding, we do get a somewhat more explicit representation of the constant c_* than what has been available so far; see Remark 6.7.)

The absence of explicit expressions for the law of the maximum can presumably be blamed on strong correlations between the spatial positions of the large local maxima. To demonstrate the point, consider an analytic bijection $f: D \rightarrow D'$ and let $\{(x_i, h_i, \phi_i): i \in \mathbb{N}\}$ enumerate the sample points of the (full) limit process η^D . Theorem 2.1 above and Theorem 2.5 of Biskup and Louidor [11] show that the point process with “points”

$$\left\{ (f(x_i), h_i + 2\sqrt{g} \log |f'(x_i)|, \phi_i) : i \in \mathbb{N} \right\} \quad (2.15)$$

is equidistributed to $\eta^{D'}$. However, the shift $2\sqrt{g} \log |f'(x_i)|$ will generally permute the order of near-maximal points and so an explicit, autonomous expression for the law of the maximum alone is not reasonable to expect.

2.3 Freezing and Liouville measure in the glassy phase.

The control of the cluster distribution in Theorem 2.3 presents us with an opportunity to resolve a couple of questions that have been debated in the spin-glass literature for some time. Before we formulate these precisely, let us give a bit of necessary motivation.

Given a sample h of the DGFF on D_N , it is natural to consider a continuous-time (variable speed) random walk on D_N that makes steps as the ordinary simple symmetric random walk but with exponential holding times whose parameter at vertex x is $e^{\beta h(x)}$. The stationary law of this walk is then given by the Gibbs (probability) measure on D_N defined by

$$\mu_{\beta,N}^D(\{x\}) := \frac{1}{\mathcal{Z}_N(\beta)} e^{\beta h_x} \quad \text{where} \quad \mathcal{Z}_N(\beta) := \sum_{x \in D_N} e^{\beta h_x}. \quad (2.16)$$

Disregarding the conventional minus sign in the exponent, the parameter $\beta \in [0, \infty)$ thus earns the meaning of the inverse temperature.

Obviously, $\mu_{\beta,N}^D$ puts the more weight on a vertex the larger the field is there. However, large field values are increasingly sparse and so a trade-off with entropy occurs. As observed by Carpentier and Le Doussal [18], this results in a phenomenon akin to that known from the Random Energy Model: The mass of $\mu_{\beta,N}^D$ asymptotically concentrates on the level set

$$\left\{ x \in D_N : h_x \approx \frac{\beta \wedge \beta_c}{\beta_c} 2\sqrt{g} \log N \right\}, \quad (2.17)$$

where $\beta_c := \alpha$. (A proof of this can be extracted directly from Daviaud's work [22].) In particular, a phase transition occurs in this model as β varies through β_c : Indeed, at $\beta = \beta_c$ the support of $\mu_{\beta,N}^D$ reaches the absolute maximum of h remains concentrated there for all $\beta > \beta_c$.

Our focus here is the detailed structure of the scaled limiting measure in the supercritical “glassy” regime; i.e., when $\beta > \beta_c$. Given a (Borel) probability measure Q on \mathbb{C} and a parameter $s > 0$, define the point measure $\Sigma_{s,Q}$ by

$$\Sigma_{s,Q}(dx) := \sum_{i \in \mathbb{N}} q_i \delta_{X_i}, \quad (2.18)$$

where $\{q_i\}$ enumerates the sample points of a Poisson process on $[0, \infty)$ with intensity $x^{-1-s} dx$ and $\{X_i\}$ are independent samples from Q , independent of the $\{q_i\}$. We will in fact need to take Q random; in this case Q is sampled first and the construction of $\Sigma_{s,Q}$ is performed conditionally on the sample of Q . Recall the notation

$$\hat{Z}^D(A) := \frac{Z^D(A)}{Z^D(D)}. \quad (2.19)$$

Then we have:

Theorem 2.6 (Liouville measure in the glassy phase) *Given $D \in \mathfrak{D}$, let D_N and m_N be as above and let Z^D denote the random measure from Theorem 2.1. For each $\beta > \beta_c := \alpha$ there is a constant $c(\beta) \in (0, \infty)$ such that*

$$\sum_{z \in D_N} e^{\beta(h_z - m_N)} \delta_{z/N}(dx) \xrightarrow[N \rightarrow \infty]{\text{law}} c(\beta) Z^D(D)^{\beta/\beta_c} \Sigma_{\beta_c/\beta, \hat{Z}^D}(dx), \quad (2.20)$$

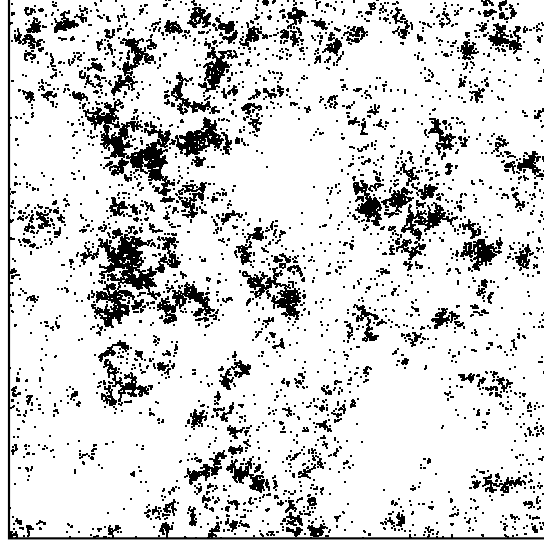


FIG. 3: A sample of the level set for the DGFF on a 500×500 square corresponding to values above $1/3$ of the absolute maximum (which occurs at height 8.17 in this sample). The fractal, and highly-correlated, nature of this set is quite apparent. Level sets for higher cutoffs become increasingly sparse and thus difficult to visualize.

where, we recall, Z^D is sampled first and $\Sigma_{\beta_c/\beta, \widehat{Z}^D}$ is defined conditionally on Z^D . Moreover, the constant $c(\beta)$ admits the explicit representation

$$c(\beta) := \beta^{-\beta/\beta_c} [E_v(Y^\beta(\phi)^{\beta_c/\beta})]^{\beta/\beta_c} \quad \text{with} \quad Y^\beta(\phi) := \sum_{x \in \mathbb{Z}^2} e^{-\beta\phi_x}. \quad (2.21)$$

In particular, $E_v(Y^\beta(\phi)^{\beta_c/\beta}) < \infty$ for each $\beta > \beta_c$.

This result settles Conjecture 6.1 of Rhodes and Vargas [34] for the DGFF on the square lattice. A proof of this conjecture has previously been given in the context of continuum (the so-called star-scale invariant) fields and the associated multiplicative chaos; cf Theorem 2.8 in Madule, Rhodes and Vargas [33]. However (as stated in [33]) the cut-off procedures employed in [33] would not permit extensions to the DGFF discussed here.

A direct consequence of Theorem 2.6 is the following estimate on the size of the level sets for samples from v :

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\{x \in \mathbb{Z}^2 : \phi_x \leq t\}| \leq \alpha, \quad v\text{-a.s.}, \quad (2.22)$$

meaning, in short, that the set where $\phi_x \leq t$ has asymptotically at most $e^{(\alpha+o(1))t}$ vertices. We in fact know that the limit exists with equality (to α) on the right-hand side, but this needs more than just the argument above. An estimate on the diameter of the level set can be gleaned from Proposition 5.8, although we do not believe that estimate to be even close to sharp.

Theorem 2.6 directly yields a limit law for the normalization constant in (2.16),

$$\forall \beta > \beta_c: \quad \mathcal{Z}_N(\beta) e^{-\beta m_N} \xrightarrow[N \rightarrow \infty]{\text{law}} Z(D)^{\beta/\beta_c} X, \quad (2.23)$$

where X is independent of Z^D and has the law of a totally skewed (a.s. positive) β_c/β -stable random variable with an explicit overall normalization. More interestingly, Theorem 2.6 also gives us the desired characterization of the above Gibbs measure $\mu_{\beta,N}^D$ for $\beta > \beta_c$. Recall that the *Poisson-Dirichlet law* with parameter $s \in (0, 1)$, to be denoted $\text{PD}(s)$, is a probability measure on non-increasing non-negative normalized sequences,

$$\left\{ \{p_i\} : p_1 \geq p_2 \geq \dots \geq 0, \sum_{i \in \mathbb{N}} p_i = 1 \right\} \quad (2.24)$$

obtained by taking the sample points of the Poisson process on $[0, \infty)$ with intensity $x^{-1-s}dx$, normalizing them by their (a.s.-finite) sum and ordering the values decreasingly. Then we have:

Corollary 2.7 (Poisson-Dirichlet limit for the Gibbs measure) *Let $D \in \mathfrak{D}$ and let $\mu_{\beta,N}^D$ be the Gibbs measure defined in (2.16). Then for all $\beta > \beta_c := \alpha$,*

$$\sum_{z \in D_N} \mu_{\beta,N}^D(\{z\}) \delta_{z/N}(dx) \xrightarrow[N \rightarrow \infty]{\text{law}} \sum_{i \in \mathbb{N}} p_i \delta_{X_i}, \quad (2.25)$$

where $\{X_i\}$ are (conditionally on Z^D) i.i.d. with common law \hat{Z}^D , while $\{p_i\} \stackrel{\text{law}}{=} \text{PD}(\beta_c/\beta)$ is independent of Z^D and thus also $\{X_i\}$.

A version of the Poisson-Dirichlet convergence (2.25) for overlap distributions has previously been established by Arguin and Zindy [9].

Another consequence of Theorem 2.6 is the proof of the so-called *freezing phenomenon*. This is a term introduced in the context of the Branching Brownian Motion by Derrida and Spohn [24] and further expounded on by Fyodorov and Bouchard [28]. Recently, Subag and Zeitouni [36] offered a deeper insight into the connection between this concept and the type of cluster process we establish in Theorem 2.1. Our result is as follows:

Corollary 2.8 (Freezing) *Let $D \in \mathfrak{D}$ and, in accord with the above references, denote*

$$G_{N,\beta}(t) := E \left(\exp \left\{ -e^{-\beta t} \sum_{x \in D_N} e^{\beta h_x} \right\} \right). \quad (2.26)$$

Let m_N be as above and let Z^D be the measure from Theorem 2.1. Then for each $\beta > \beta_c := \alpha$ there is a constant $\tilde{c}(\beta) \in \mathbb{R}$ such that

$$G_{N,\beta}(t + m_N + \tilde{c}(\beta)) \xrightarrow[N \rightarrow \infty]{} E(e^{-Z^D(D)} e^{-\alpha t}). \quad (2.27)$$

The constant $\tilde{c}(\beta)$, given explicitly in (6.70), depends only on the law of v and that only via the expectation in (2.21).

A non-degenerate limit of $G_{N,\beta}(t + m_{N,\beta})$ — with a suitable centering sequence $m_{N,\beta}$ — is expected to exist for all values of $\beta \in [0, \infty)$. The term “freezing” then refers to the fact that the limit function ceases to depend on β (i.e., “freezes”) once β passes through β_c . Notwithstanding, the existence of the limit for $\beta \leq \beta_c$ remains, to our knowledge, an open problem.

2.4 Heuristics and outline.

The rest of this article is devoted to the proofs of the above results. In order to give an outline of what is to come, let us begin by an appealing heuristic argument why the above v should appear as the distribution of the clusters.

A natural way to get to the clusters is by conditioning on the relevant local maxima. Assuming these occur at points x_1, \dots, x_n with the field values at $m_N + t_1, \dots, m_N + t_n$, respectively, if we condition h on just taking these values at these points, the field decomposes into the sum

$$h^{D_N \setminus \{x_1, \dots, x_n\}} + \mathfrak{g}_N, \quad (2.28)$$

where $h^{D_N \setminus \{x_1, \dots, x_n\}}$ is now the DGFF in $D_N \setminus \{x_1, \dots, x_n\}$ while \mathfrak{g}_N is discrete harmonic there, equal to $m_N + t_i$ at each x_i and vanishing outside D_N . Since x_1, \dots, x_n will be separated by distances of order N with high probability, as $N \rightarrow \infty$, we have $m_N + t_i - \mathfrak{g}_N(x) \rightarrow \frac{2}{\sqrt{g}} \mathfrak{a}(x - x_i)$ whenever x is sufficiently near x_i . Hence,

$$(m_N + t_i) - \left[h^{D_N \setminus \{x_1, \dots, x_n\}}(x_i + \cdot) + \mathfrak{g}_N(x_i + \cdot) \right] \xrightarrow[N \rightarrow \infty]{\text{law}} \phi^{(i)}(\cdot) + \frac{2}{\sqrt{g}} \mathfrak{a}(\cdot), \quad (2.29)$$

where $\phi^{(i)} \stackrel{\text{law}}{=} -\phi^{(i)}$ is distributed according to ν^0 . However, once we impose that each x_i is also a local maximum, the field in (2.28) must also not exceed the value at x_i in an r -neighborhood of x_i . This forces the conditioning on $\phi^{(i)} + \frac{2}{\sqrt{g}} \mathfrak{a} \geq 0$ in $\Lambda_r(x_i)$. See Fig. 4 for an illustration.

Our proof of Theorem 2.1 proceeds more or less along these lines, albeit not without a significant amount of technical overhead caused, in hindsight, by the singular nature of the conditioning spelled out in Theorem 2.4. Indeed, in order to prove the Poisson law in (2.4), we have to establish a version of (2.29) with $\phi^{(1)}, \dots, \phi^{(n)}$ independent of each other. This is easy for the unconditioned law but becomes a challenge once we condition on small probability events.

The approach we take is that we first focus on a single local maximum and analyze the situation around it in full detail. This is facilitated by a natural *concentric decomposition* of the DGFF pinned to a high value (of the form $m_N + t$) into a sum of independent and, more or less, localized random fields with good control on the tails. The development and basic properties of the concentric decomposition are the subject of Section 3.

An attractive feature of the concentric decomposition is that the overall growth rate of the pinned field can be encoded into a “backbone” random walk (for the pinning field in all of \mathbb{Z}^2) or an associated *random-walk bridge* (for the pinning problem in finite volume). The requirement that the field be maximized at the pinning point then more or less amounts to having this random walk/bridge stay above a polylogarithmic curve for a large interval of times; see Section 4.1.

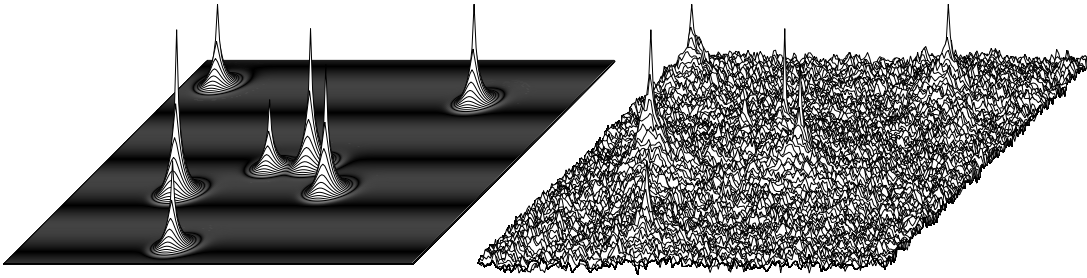


FIG. 4: Left: An illustration of function \mathfrak{g}_N from (2.28) for the underlying domain D being a unit square, $N := 200$ and $n := 7$ local maxima being fixed. Right: A sample of the full field $h^{D_N \setminus \{x_1, \dots, x_n\}} + \mathfrak{g}_N$ conditioned on the values at x_1, \dots, x_7 to be local maxima in an r -neighborhood thereof for $r := 20$.

Estimating the probability that the random walk/bridge stays above such curves is easier to perform first for Brownian motion/bridge; the statements (Propositions 4.7–4.10) are to be found in Section 4.2 with the proofs relegated to Appendix A. Straightforward interpolation arguments, simplified considerably by the fact that the random walk/bridge has Gaussian (albeit not identically distributed) steps, then allow us to pull these to the discrete time setting; see Section 4.3.

The underlying mechanism that makes all this work is *entropic repulsion*, which causes a random walk conditioned to stay above a slowly-varying negative curve to actually rise, with overwhelming probability, even above a (slowly-varying) positive curve. Explicit statements appear in Propositions 4.13–4.14. In Section 4.4 we use this to derive a number of useful estimates for the various attributes of the concentric decomposition; these feed repeatedly into the proofs later. It is Section 4 where our paper makes close contact with the literature on the Branching Brownian Motion; specifically, the pioneering work by Bramson [14].

In Section 5 we return to the problem of the pinned DGFF and start harvesting results. First (in Section 5.2) we establish the existence and non-degeneracy of the cluster law and thus prove Theorem 2.3. With some additional work (spelled out in Sections 5.3–5.4), this yields also the proof of the full scaling limit in Theorem 2.1. The main technical input here is the conditional version of the convergence in (2.29), still for a single point, carried out in Propositions 5.1–5.2. The contributions of individual local maxima are separated with the help of Proposition 5.10. These propositions are the core technical steps of the proofs of our main results.

Section 6 addresses the proofs of the remaining theorems; first the local limit theorem for the position and value of the maximum (Theorem 2.5) and then the proof of Theorem 2.6 and Corollaries 2.7–2.8 dealing with the Liouville measure, Poisson-Dirichlet statistics and freezing. A key point here is the fact that the contribution from the clusters to the measure on the left-hand side of (2.20) can be completely absorbed, via the expectation of the quantity $Y^\beta(\phi)$ in (2.21), into the overall normalizing constant $c(\beta)$. This can (roughly) be attributed to:

Observation 2.9 *Let $\{z_i: i \in \mathbb{N}\}$ enumerate the sample points from a Gumbel Poisson point process with intensity $e^{-\lambda x}dx$ for some $\lambda > 0$ and let $\{X_i: i \in \mathbb{N}\}$ be i.i.d. random variables with $\theta := E(e^{\lambda X_1}) < \infty$, independent of $\{z_i: i \in \mathbb{N}\}$. Then also $\{z_i + X_i: i \in \mathbb{N}\}$ is a sample from a Gumbel process but this time with intensity $\theta e^{-\lambda x}dx$.*

In fact, absorbing the contribution of the clusters into a single random variable is the main step of the above proofs; the rest follows fairly directly (by exponentiating) from Theorem 2.1.

Our proofs naturally use a number of facts about Gaussian processes, and in particular the DGFF, that have been proved earlier. To systematize referencing, we list these results as separate lemmas in Appendix B and then quote only these lemmas in the proofs.

2.5 Connections and open questions.

We conclude this section by listing some questions of further interest. Our first question concerns analytic properties of the density ρ^D from Theorem 2.6. The asymptotic expression (2.13) and our simulations in Fig. 2 suggest the following:

Conjecture 2.10 *For each $t \in \mathbb{R}$, the function $x \mapsto \rho^D(t, x)$ is bounded and tends to zero as x approaches ∂D . In particular, the function admits a continuous extension to all of \bar{D} .*

Notice that, although the considerations of the DGFF (cf Lemma B.12) give the absolute continuity of the measure in (2.12) with respect to the Lebesgue measure, they only imply a certain integrability condition for $x \mapsto \rho^D(t, x)$. Unfortunately, our proofs do not seem to be able to determine the boundary regularity of this function either.

As our next question, we wish to point out that the limits in Theorem 2.6 and Corollary 2.7 should have versions even for $\beta = \beta_c$:

Conjecture 2.11 *There is $c \in (0, \infty)$ such that for each $D \in \mathfrak{D}$,*

$$\sqrt{\log N} \sum_{z \in D_N} e^{\beta_c(h_z - m_N)} \delta_{z/N}(\mathrm{d}x) \xrightarrow[N \rightarrow \infty]{\text{law}} c Z^D(\mathrm{d}x). \quad (2.30)$$

In particular,

$$\sum_{z \in D_N} \mu_{\beta_c, N}^D(\{z\}) \delta_{z/N}(\mathrm{d}x) \xrightarrow[N \rightarrow \infty]{\text{law}} \widehat{Z}^D(\mathrm{d}x). \quad (2.31)$$

A version of the convergence in (2.30) is claimed in Rhodes and Vargas [34, Theorem 5.13] in the framework of isoradial graphs, although the details given there (for the relevant critical case) are scarce. A full proof of (2.31) still requires identification of the limit measure there with our Z^D which has not been accomplished so far. We note that, unlike for $\beta > \beta_c$ where the limit measures in (2.20) and (2.25) are purely atomic, the limit laws in (2.30–2.31) have no atoms at all (see Biskup and Louidor [11, Theorem 2.1]).

A corresponding question arises also for $\beta < \beta_c$, although there the limit of the Liouville measure is already reasonably well understood (see Rhodes and Vargas [34, Theorem 5.12 and Appendix B]). However, it is not clear how this translates into the control of the level sets in (2.17) when “ \approx ” is replaced by “ \geq ” and m_N is replaced by a function that grows in the leading order as $\zeta(2\sqrt{g} \log N)$ for some $\zeta \in (0, 1)$. Here is an attempt to formulate this more precisely:

Question 2.12 *Let $\zeta \in (0, 1)$ and let D_N arise from a $D \in \mathfrak{D}$ as above. Is there a_N depending only on D and ζ such that*

$$\frac{1}{a_N} \sum_{x \in D_N} \delta_{x/N} \otimes \delta_{h_N - 2\sqrt{g}\zeta \log N} \quad (2.32)$$

converges in law as $N \rightarrow \infty$ to a non-trivial random measure?

The point of this question is to find out if one can capture the scaling limit of the level set of the form (2.17), including local information, using a meaningful continuum object. Continuum counterparts of this question exist; e.g., based on the notion of a *thick point* of the (continuum) Gaussian Free Field analyzed by Hu, Miller and Peres [29].

A natural question is whether and how our results extend to the class of logarithmically correlated Gaussian fields in general dimensions $d \geq 1$. Here the convergence of the law of the centered maximum has already been proved (Ding, Roy and Zeitouni [23]) but, for the lack the Gibbs-Markov property in $d \neq 2$, our techniques do not apply. We remark that the scaling limit of the extremal process of the DGFF in $d \geq 3$, which is *not* logarithmically correlated, has been shown to coincide, modulo an overall shift, with that of i.i.d. Gaussian random variables (Chiarini, Cipriani, Hazra [19–21]).

Another interesting direction concerns the corresponding problem for various non-Gaussian models on \mathbb{Z}^2 with fluctuation structure described, at large scales, by the Gaussian Free Field. This includes the gradient models with uniformly-strictly convex potentials or the local time of

the simple random walk run for multiples of the cover time. Unfortunately, here even the tightness of the maximum remains open. Notwithstanding, for the local time associated with the simple random walk on a homogeneous tree, the fluctuations are, at large scales, those of a Gaussian Branching Random Walk. In this case, Abe [1] was able to show that the convergence of the type (1.6) holds for a suitably defined process of local maxima of the local time.

3. FIELD PINNED TO A HIGH VALUE

We are now ready to start the exposition of the proofs. Here we show that a pinned DGFF naturally decomposes into the sum of independent random fields indexed by a sequence of nested domains and use it to represent the growth rate of the field by way of a random walk.

Throughout this whole section, D (or similar letters) will denote a generic finite set $D \subset \mathbb{Z}^2$ while D_N will keep denoting the set as in (2.1–2.2) for some (notationally implicit) underlying continuum domain. We will write h^D (instead of just h) to denote the DGFF in D and will write h_x^D or $h^D(x)$ to denote its value at x . The arguments use various standard facts about the DGFF and harmonic analysis on \mathbb{Z}^2 ; these are for reader's convenience collected in Appendix B.

3.1 Simplifying the conditioning.

We begin by a reduction argument. Recall that our ultimate goal is to control the position and field value at, and the “shape” of the configuration around, the nearly-maximal local maxima. Thanks to the Gibbs-Markov property (Lemma B.6) and estimates on separation of near-maximal values (Lemma B.11), it will suffice to do this just for one local maximum. The main task is thus the $N \rightarrow \infty$ asymptotic of the probability

$$P\left(h^{D_N}(0) - h^{D_N} \in A: h^{D_N} \leq m_N + t + s, h^{D_N}(0) \geq m_N + t\right) \quad (3.1)$$

for $t \in \mathbb{R}$, $s \geq 0$ and events A that depend only on a finite number of coordinates near 0. (Here and henceforth, $h \leq f$ means that $h(x) \leq f(x)$ on the natural domain of h .) We will instead study the conditional probability

$$P\left(h^{D_N}(0) - h^{D_N} \in A, h^{D_N} \leq m_N + t + s \mid h^{D_N}(0) = m_N + t\right). \quad (3.2)$$

This is sufficient for (3.1) because the probability (density) of the conditional event is explicitly available and (3.1) can thus be obtained from (3.2) by integrating over t .

Remark 3.1 The type of “singular” conditioning as in (3.2) will be used frequently throughout the rest of this paper. In all such cases it will be clear that the conditional random variable has a well-defined, continuous probability density with respect to the Lebesgue measure and so the conditioning boils down to substituting the conditional value for that variable.

In order to control (3.2) note that, due to the Gaussian nature of the field, conditioning on a large value at a given point can be reduced to a shift in the mean. More precisely, given a finite $D \subset \mathbb{Z}^2$ with $0 \in D$ let $\mathfrak{g}^D: \mathbb{Z}^2 \rightarrow [0, 1]$ denote the (unique) function such that

- (1) \mathfrak{g}^D is discrete harmonic on $D \setminus \{0\}$,
- (2) $\mathfrak{g}^D(0) = 1$ and $\mathfrak{g}^D(x) = 0$ for $x \notin D$.

By the maximum principle, \mathfrak{g}^D takes values in $[0, 1]$. The conditioning then simplifies as:

Lemma 3.2 Suppose $D \subset \mathbb{Z}^d$ is finite with $0 \in D$. Then for all $t, s \in \mathbb{R}$ and any event A ,

$$P\left(h^D \in A, h^D \leq s \mid h^D(0) = t\right) = P\left(h^D + t\mathbf{g}^D \in A, h^D \leq s - t\mathbf{g}^D \mid h^D(0) = 0\right). \quad (3.3)$$

Proof. By the Gibbs-Markov property (cf Lemma B.6), we have

$$h^D \stackrel{\text{law}}{=} h^D(0)\mathbf{g}^D + h^{D \setminus \{0\}}, \quad (3.4)$$

where the two fields on the right are regarded as independent. This formula shows that $h^{D \setminus \{0\}}$, the DGFF in $D \setminus \{0\}$, is also the DGFF in D conditioned on $h^D(0) = 0$. Plugging these facts into the left-hand side (3.3), the claim follows. \square

Most of the time we will use this lemma with $s := t$ — which amounts to dealing with absolute maximum — but the above formulation will be useful when we want to address local maxima as well; e.g., in Proposition 6.8 and Lemma 6.9.

3.2 Concentric decomposition.

Having reduced the problem to a field pinned to zero, the next technical step is a representation of this field as a sum of independent random fields. The pinning at a single point naturally leads us to consider a decomposition along a nested sequence of domains. Since we will ultimately work with ℓ^∞ -balls centered at the origin, we will refer to this as a *concentric decomposition*.

Given a set $B \subset \mathbb{Z}^2$, let ∂B denote the set of vertices on its external boundary. Consider an increasing sequence of connected sets $\Delta^0, \Delta^1, \dots, \Delta^n \subset \mathbb{Z}^2$ satisfying

$$\Delta^0 := \{0\} \quad \text{and} \quad \overline{\Delta^k} := \Delta^k \cup \partial \Delta^k \subseteq \Delta^{k+1}, \quad k = 0, \dots, n-1. \quad (3.5)$$

Recalling the notation h^{Δ^n} for the DGFF in Δ^n , we now define

$$\varphi_k(x) := \begin{cases} E(h^{\Delta^n}(x) \mid \sigma(h^{\Delta^n}(z) : z \in \partial \Delta^{k-1} \cup \partial \Delta^k)) \\ \quad - E(h^{\Delta^n}(x) \mid \sigma(h^{\Delta^n}(z) : z \in \partial \Delta^k)) , & k = 1, \dots, n, \\ h^{\Delta^n}(x) - E(h^{\Delta^n}(x) \mid \sigma(h^{\Delta^n}(z) : z \neq 0)), & k = 0, \end{cases} \quad (3.6)$$

and let

$$\chi_k(x) := \varphi_k(x) - E(\varphi_k(x) \mid \sigma(\varphi_k(0))), \quad k = 0, \dots, n. \quad (3.7)$$

Then we set

$$h'(x) := h^{\Delta^n}(x) - \sum_{k=0}^n \varphi_k(x) \quad (3.8)$$

and let

$$h'_k(x) := h'(x) 1_{\Delta^k \setminus \Delta^{k-1}}(x), \quad k = 0, \dots, n. \quad (3.9)$$

All the fields above are defined on the same probability space as h^{Δ^n} and all can be regarded as fields on all of \mathbb{Z}^2 . Their distributional properties are summarized in:

Proposition 3.3 (Concentric decomposition) Suppose the domains $\{\Delta^k : k = 0, \dots, n\}$ obey the restrictions in (3.5). Then the following holds: The random objects in the union

$$\{\varphi_k(0) : k = 0, \dots, n\} \cup \{\chi_k : k = 0, \dots, n\} \cup \{h'_k : k = 0, \dots, n\} \quad (3.10)$$

are all independent of one another. Their individual laws are (multivariate) normal and they are determined by the following properties:

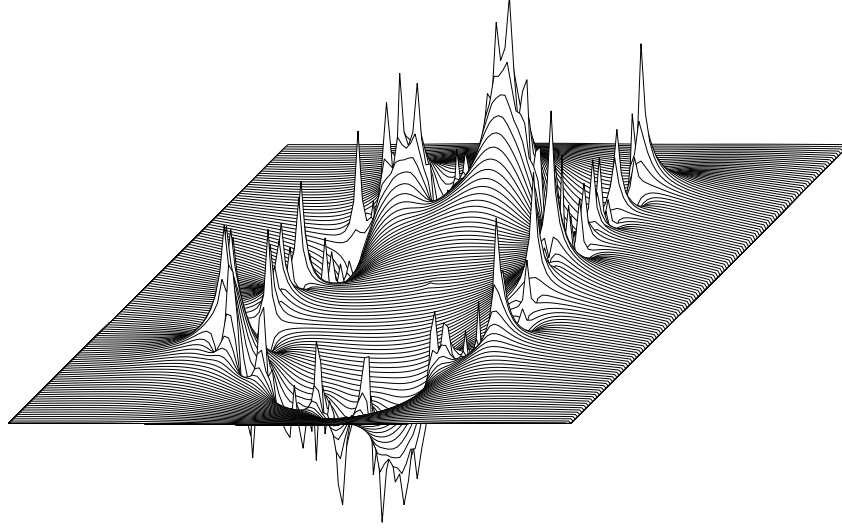


FIG. 5: The graph of a random sample of χ_k from (3.7) for Δ^k being a square in \mathbb{Z}^2 of side-length 128. The function φ_k from which χ_k is derived takes values of the DGFF h^{Δ^k} on $\partial\Delta^{k-1}$ and is discrete harmonic elsewhere; χ_k generally fails to be discrete harmonic also at the center of Δ^k due to the second term in (3.7).

(1) For $k = 1, \dots, n$, the field in (3.6) obeys

$$\varphi_k \stackrel{\text{law}}{=} E(h^{\Delta^k} | \sigma(h^{\Delta^k} : z \in \partial\Delta^{k-1})), \quad k = 1, \dots, n, \quad (3.11)$$

while, for $k = 0$,

$$\varphi_0 = \varphi_0(0)1_{\{0\}}, \quad \text{where} \quad \varphi_0(0) \stackrel{\text{law}}{=} \mathcal{N}(0, 1). \quad (3.12)$$

A.e. sample path of φ_k is discrete harmonic on $\Delta^k \setminus \partial\Delta^{k-1}$ and zero on $\mathbb{Z}^2 \setminus \Delta^k$. The law of χ_k is then determined from (3.7).

(2) For $k = 1, \dots, n$, for the fields in (3.9) we have

$$h'_k \stackrel{\text{law}}{=} h^{\Delta^k \setminus \overline{\Delta^{k-1}}} \quad (3.13)$$

while $h'_0 = 0$. In particular, h' in (3.8) has the law of the DGFF in $\bigcup_{k=1}^n \Delta^k \setminus \overline{\Delta^{k-1}}$ with, per our convention, zero boundary conditions outside of this set.

In addition, we have $\text{Var}(\varphi_k(0)) > 0$ for all $k = 0, \dots, n$ and, letting

$$\mathfrak{b}_k(x) := \frac{1}{\text{Var}(\varphi_k(0))} E\left(\varphi_k(0)(\varphi_k(x) - \varphi_k(0))\right), \quad (3.14)$$

the following representation holds

$$h^{\Delta^n}(x) = \sum_{k=0}^n (1 + \mathfrak{b}_k(x)) \varphi_k(0) + \sum_{k=0}^n \chi_k(x) + \sum_{k=0}^n h'_k(x). \quad (3.15)$$

Proof. Fix $n \geq 0$ and consider the σ -algebras

$$\mathcal{F}_k := \sigma\left(h^{\Delta^n}(z) : z \in \bigcup_{\ell=n-k}^n \partial\Delta^\ell\right), \quad k = 0, \dots, n+1, \quad (3.16)$$

where $\partial\Delta^{-1} := \Delta^0$. Obviously, $k \mapsto \mathcal{F}_k$ is non-decreasing. Since h^{Δ^n} is fixed to zero on $\partial\Delta^n$, we have $E(h^{\Delta^n}(x)|\mathcal{F}_0) = E(h^{\Delta^n}(x)) = 0$ and so

$$\varphi(x) := E(h^{\Delta^n}(x)|\mathcal{F}_{n+1}) = \sum_{k=0}^n \left(E(h^{\Delta^n}(x)|\mathcal{F}_{k+1}) - E(h^{\Delta^n}(x)|\mathcal{F}_k) \right). \quad (3.17)$$

We claim that

$$E(h^{\Delta^n}(x)|\mathcal{F}_{k+1}) - E(h^{\Delta^n}(x)|\mathcal{F}_k) = \varphi_{n-k}(x). \quad (3.18)$$

Indeed, the Gibbs-Markov property (Lemma B.6) tells us that $x \mapsto E(h^{\Delta^n}(x)|\mathcal{F}_k)$ is the discrete-harmonic extension of the restriction of h^{Δ^n} to the set in the definition of \mathcal{F}_k . The two terms on the left of (3.18) have equal extensions outside Δ^{n-k} and so their difference vanishes there, while inside Δ^{n-k} it is equal to the difference between the harmonic extension of h^{Δ^n} restricted to $\partial\Delta^{n-k} \cup \partial\Delta^{n-k-1}$ and the harmonic extension of h^{Δ^n} restricted to $\partial\Delta^{n-k}$. Comparing with (3.6), this yields (3.18).

Next we observe that the terms in the sum (3.17) are independent of one another because they are uncorrelated (being martingale increments) Gaussians. By the Gibbs-Markov property again, h' in (3.8) has the law of the DGFF in $\bigcup_{k=1}^n \Delta^k \setminus \overline{\Delta^{k-1}}$. Since the DGFFs in disconnected sets are independent, we get

$$h^{\Delta^n}(x) = \sum_{k=0}^n \varphi_k(x) + \sum_{k=0}^n h'_k \quad (3.19)$$

with all the fields on the right independent of one another. The Gibbs-Markov property then also helps us check (3.11–3.12).

Now define χ_k by (3.7). Then $\varphi_k(0)$ and χ_k are uncorrelated and thus independent, yielding the claimed independence of the family (3.10). Interpreting conditioning as a projection gives

$$E(\varphi_k(x)|\sigma(\varphi_k(0))) = \mathfrak{f}_k(x)\varphi_k(0) \quad (3.20)$$

for some deterministic function \mathfrak{f}_k . Writing \mathfrak{f}_k as $1 + \mathfrak{b}_k$, a covariance calculation shows that \mathfrak{b}_k must be given by (3.14) provided we can verify $\text{Var}(\varphi_k(0)) > 0$. This follows from

$$\text{Var}(\varphi_k(0)) = \text{Var}(h^{\Delta^k}(0)) - \text{Var}(h^{\Delta^{k-1}}(0)) \quad (3.21)$$

and the connectedness of Δ^k along with $\Delta^{k-1} \subsetneq \Delta^k$, as implied by (3.5). (Indeed, by (1.1) the variances are the expected numbers of returns of the simple random walk from 0 to 0 before the walk leaves the given set and, under the stated conditions, there is a finite path of the walk from 0 to 0 that stays in Δ^k but leaves Δ^{k-1} along the way.) \square

The decomposition (3.15) yields:

Corollary 3.4 *Assume $\{\varphi_k(0) : k \geq 0\}$, $\{\chi_k : k \geq 0\}$ and $\{h'_k : k \geq 0\}$ are independent objects with the laws as specified in Proposition 3.3 and let $\{\mathfrak{b}_k : k \geq 0\}$ be as in (3.14). Then, for each $n \geq 0$, the sum on the right-hand side of (3.15) has the law of the DGFF in Δ^n . In particular, the whole family $\{h^{\Delta^n} : n \geq 0\}$ can be constructed on a single probability space by imposing (3.15) for all $n \geq 0$. Moreover, the object defined in (3.6) is then given by*

$$\varphi_k(x) = (1 + \mathfrak{b}_k(x))\varphi_k(0) + \chi_k(x) \quad (3.22)$$

for all $k \geq 0$ and all $x \in \mathbb{Z}^2$.

Proof. As is easily checked, the laws of the random variables $\varphi_k(0)$ and random fields χ_k and h'_k as well as the function b_k depend only on Δ^{k-1} and Δ^k and, in particular, have no explicit dependence on n . Hence, (3.15) can be imposed simultaneously for all $n \geq 0$. The final argument in the proof of Proposition 3.3 then yields (3.22). \square

We will henceforth assume that all random variables in (3.10) are realized on the same probability space (as independent with laws as above) and the fields $\{h^{\Delta^n} : n \geq 0\}$, resp., $\{\varphi_n : n \geq 0\}$ are derived from these via (3.15), resp., (3.22).

The sequence $\{\varphi_k(0) : k \geq 0\}$ will be central to our arguments and that particularly so via the sequence of its partial sums

$$S_k := \sum_{\ell=0}^{k-1} \varphi_\ell(0), \quad k \geq 0. \quad (3.23)$$

Here it is important to note that, in light of

$$b_k(0) = 0 \quad \text{as well as} \quad \chi_k(0) = 0 \quad \text{and} \quad h'_k(0) = 0 \quad \text{a.s.,} \quad k \geq 0, \quad (3.24)$$

we have $S_k = h^{\Delta^{k-1}}(0)$ for all $k \geq 1$. The independence of $\{\varphi_k(0) : k \geq 0\}$ suggests to regard $\{S_k : k \geq 1\}$ as a path of a *random walk*, albeit with time-inhomogeneous steps. For the conditional event in (3.3) we then get

$$h^{\Delta^n}(0) = 0 \quad \Leftrightarrow \quad S_{n+1} = 0. \quad (3.25)$$

The conditioning on $h^{\Delta^n}(0) = 0$ thus amounts to the requirement that the random walk be back to its starting point after $n+1$ steps.

Remark 3.5 The above decomposition of the DGFF can be regarded as a concentric analogue of the finite-range decomposition of the (homogeneous) Gaussian Free Field (discrete or continuum) used in renormalization-based treatments of interacting random fields (Brydges [17]). There is also some vague analogy between $\{S_n\}$ and the random walk that arises in the spine decomposition of a Branching Random Walk (e.g., Aïdékon [4]).

3.3 Useful estimates.

Our next task is to represent the overall growth of h^{Δ^n} in terms of the random walk (3.23). This will require rather tight control of the various terms on the right-hand side of (3.15) and, for convenience, a specific choice of the domains in (3.5). Therefore, we will set

$$\Delta^k := \{x \in \mathbb{Z}^2 : |x|_\infty \leq 2^k\}, \quad k \geq 1, \quad (3.26)$$

for the remainder of this paper. (Generalizations when Δ^n is replaced by a more general domain of the same scale will be discussed in Section 4.5.) Our discussion of the behavior of the objects entering (3.15) starts with the random variables $\varphi_k(0)$:

Lemma 3.6 *We have*

$$\inf_{k \geq 1} \text{Var}(\varphi_k(0)) > 0. \quad (3.27)$$

Moreover,

$$\lim_{k \rightarrow \infty} \text{Var}(\varphi_k(0)) = g \log 2. \quad (3.28)$$

Proof. The strict positivity of $\text{Var}(\phi_k(0))$ for each $k \geq 0$ was proved in Proposition 3.3 so it suffices to show (3.28). This follows from (3.21), the representation $\text{Var}(h^{\Delta^k}(0))$ as the Green function $G^{\Delta^k}(0,0)$, the definition of Δ^k and the asymptotic form $G^{\Delta^k}(0,0) = g \log(2^k) + c_0 + o(1)$ as $k \rightarrow \infty$ for some constant c_0 . (This form is derived using Lemmas B.3–B.4.) \square

Next we will address the behavior of function \mathfrak{b}_k :

Lemma 3.7 *Let $k \in \{0, 1, \dots\}$. Then \mathfrak{b}_k is discrete harmonic in $\mathbb{Z}^2 \setminus (\partial\Delta^k \cup \partial\Delta^{k-1})$ and bounded uniformly in k . In addition, we have $\mathfrak{b}_k(x) \geq -1$ for all $x \in \mathbb{Z}^2$,*

$$\mathfrak{b}_k(x) = -1, \quad x \notin \Delta^k, \quad (3.29)$$

and, for some $c \in (0, \infty)$ independent of k and “dist” denoting the ℓ^∞ -distance on \mathbb{Z}^2 ,

$$|\mathfrak{b}_k(x)| \leq c \frac{\text{dist}(0, x)}{\text{dist}(0, \partial\Delta^k)}, \quad x \in \Delta^k. \quad (3.30)$$

Proof. The stated discrete harmonicity of \mathfrak{b}_k , resp., (3.29) are directly checked from the definition in (3.14), resp., (3.11). To get that $1 + \mathfrak{b}_k \geq 0$ it suffices to check that ϕ_k has positive correlations. This follows either directly from the strong-FKG property (cf Lemma B.8) or by

$$E(\phi_k(x)\phi_k(0)) = E(h^{\Delta^k}(x)h^{\Delta^k}(0)) - E(h^{\Delta^{k-1}}(x)h^{\Delta^{k-1}}(0)) \quad (3.31)$$

combined with the argument at the end of the proof of Proposition 3.3. The uniform boundedness of \mathfrak{b}_k follows from (3.31) and Lemmas B.3–B.4.

It remains to prove (3.30). Clearly, by our choice of Δ^k and the boundedness of \mathfrak{b}_k , we only need to do this for $x \in \Delta^{k-2}$. Writing $H^D(x, y)$ — where $D \subset \mathbb{Z}^2$, $x \in D$ and $y \in \partial D$ — for the harmonic measure for the simple random walk, the harmonicity of \mathfrak{b}_k in Δ^{k-1} and $\mathfrak{b}_k(0) = 0$ yield

$$\mathfrak{b}_k(x) = \sum_{y \in \partial\Delta^{k-1}} [H^{\Delta^{k-1}}(x, y) - H^{\Delta^{k-1}}(0, y)] \mathfrak{b}_k(y). \quad (3.32)$$

By Lemma B.5, we have

$$|H^{\Delta^{k-1}}(x, y) - H^{\Delta^{k-1}}(x', y)| \leq c \frac{\text{dist}(x, x')}{\text{dist}(0, \partial\Delta^{k-1})} \frac{1}{|\partial\Delta^{k-1}|}, \quad x, x' \in \Delta^{k-2}. \quad (3.33)$$

for some constant $c \in (0, \infty)$. Using that \mathfrak{b}_k is bounded on $\partial\Delta^{k-1}$, and noting that the diameter of Δ^k is proportional to that of Δ^{k-2} , we readily infer (3.30). \square

Our next item of concern is the law of χ_k . Since χ_k is a.s. discrete harmonic on Δ^{k-1} , its control on Δ^ℓ for $\ell \leq k-2$ is fairly straightforward:

Lemma 3.8 *There is a constant $c \in (0, \infty)$ such that for all $k \in \mathbb{N}$ and all ℓ with $0 \leq \ell \leq k-2$,*

$$E\left(\max_{x \in \Delta^\ell} |\chi_k(x)|\right) \leq c 2^{\ell-k} \quad (3.34)$$

and, for some constant $c' \in (0, \infty)$ and all $\lambda > 0$,

$$P\left(\left|\max_{x \in \Delta^\ell} \chi_k(x) - E\left(\max_{x \in \Delta^\ell} \chi_k(x)\right)\right| > \lambda\right) \leq 2e^{-c' 4^{k-\ell} \lambda^2}. \quad (3.35)$$

We have $\chi_k(x) = 0$ a.s. whenever $x \notin \Delta^k$.

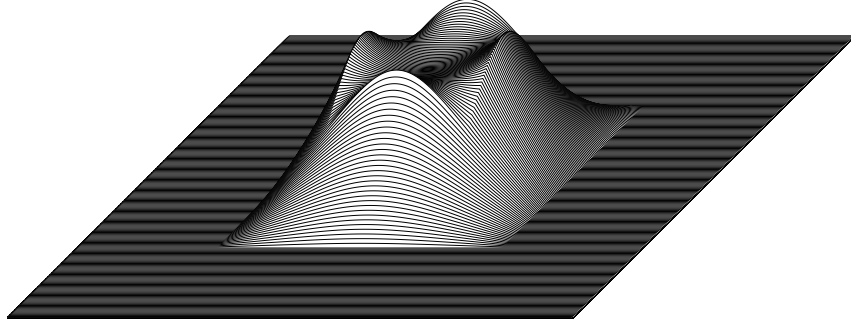


FIG. 6: The graph of function \mathfrak{b}_k for $k := 7$ (and Δ^k as in (3.26)). Notice that \mathfrak{b}_k does attain strictly positive values, but that only so in the vicinity of $\partial\Delta^{k-1}$. (The value outside Δ^k is negative one.)

Proof. We start with (3.34). The sample paths of χ_k are discrete harmonic on Δ^{k-1} and obey $\chi_k(0) = 0$. Using the notation from the previous proof, for each $x \in \Delta^{k-2}$,

$$\chi_k(x) = \sum_{z \in \partial\Delta^{k-2}} [H^{\Delta^{k-2}}(x, z) - H^{\Delta^{k-2}}(0, z)] \chi_k(z). \quad (3.36)$$

In light of (3.33), we get

$$\max_{x \in \Delta^\ell} |\chi_k(x)| \leq c 2^{\ell-k} \max_{x \in \partial\Delta^{k-2}} |\chi_k(x)|, \quad \ell = 0, \dots, k-2. \quad (3.37)$$

It thus suffices to show that the expected maximum on the right is uniformly bounded in k .

By independence of $\varphi_k(0)$ and χ_k and the stated harmonicity, for all $x, y \in \Delta^{k-1}$,

$$\begin{aligned} E(|\chi_k(x) - \chi_k(y)|^2) &\leq E(|\varphi_k(x) - \varphi_k(y)|^2) \\ &= \sum_{z, z' \in \partial\Delta^{k-1}} [H^{\Delta^{k-1}}(x, z) - H^{\Delta^{k-1}}(y, z)] [H^{\Delta^{k-1}}(x, z') - H^{\Delta^{k-1}}(y, z')] G^{\Delta^k}(z, z'). \end{aligned} \quad (3.38)$$

Invoking the standard asymptotic form of the Green function, the average of $G^{\Delta^k}(z, z')$ over the points $z, z' \in \partial\Delta^{k-1}$ is bounded uniformly in k . From (3.33) we thus conclude

$$E(|\chi_k(x) - \chi_k(y)|^2) \leq c \left(\frac{|x - y|}{2^k} \right)^2, \quad x, y \in \overline{\Delta^{k-1}} \quad (3.39)$$

with c independent of k . Using the Fernique criterion (cf Lemma B.1) with the counting measure on $\overline{\Delta^{k-2}}$ as the majorization measure, the expected maximum of χ_k on $\overline{\Delta^{k-2}}$ is found to be bounded uniformly in k . By (3.37), we get (3.34).

To get (3.35), we note that (since $\chi_k(0) = 0$) the bound (3.39) shows that the variance of $\chi_k(x)$ is bounded uniformly in $x \in \Delta^{k-1}$. By the argument leading to (3.37), we thus get

$$\max_{x \in \Delta^\ell} \text{Var}(\chi_k(x)) \leq c 2^{2(\ell-k)}, \quad \ell = 0, \dots, k-2. \quad (3.40)$$

The Borell-Tsirelson inequality (cf Lemma B.2) then yields (3.35). \square

Unfortunately, χ_k is not discrete harmonic on $\partial\Delta^{k-1} \cup \partial\Delta^k$, and so its control on $\Delta^k \setminus \Delta^{k-2}$ requires a different argument. We will rely on the fact that, by combining χ_k with h'_k we get (more or less) the DGFF in $\Delta^k \setminus \Delta^{k-2}$. Let us therefore first address the tails of h'_k .

Lemma 3.9 *Recall the notation m_N for the quantity from (1.2). There are constants $c_1, c_2 \in (0, \infty)$ such that for all $k \geq 1$ and all $\lambda > 0$,*

$$P\left(\left|\max_{x \in \Delta^k \setminus \Delta^{k-1}} h'_k(x) - m_{2^k}\right| > \lambda\right) \leq c_1 e^{-c_2 \lambda}. \quad (3.41)$$

(We have $h'_k(x) = 0$ a.s. outside the annulus $\Delta^k \setminus \Delta^{k-1}$.) As $k \rightarrow \infty$ the joint law of

$$(2^{-k} \operatorname{argmax}_{\Delta^k \setminus \Delta^{k-1}} h'_k, \max_{x \in \Delta^k \setminus \Delta^{k-1}} h'_k(x) - m_{2^k}) \quad (3.42)$$

tends to a non-degenerate distribution on $(-1, 1)^2 \times \mathbb{R}$.

Proof. A bound of the form (3.41) has been proved for square domains in Ding and Zeitouni [26]; cf Lemma B.13. Concerning the maximum in the annuli $\Delta^k \setminus \Delta^{k-1}$ we apply the bounds in Lemma B.7 along with the fact that $N \mapsto m_N$ is slowly varying and that $\Delta^k \setminus \Delta^{k-1}$ contains and is contained in a square of side of order 2^k .

The convergence in (3.42) has been proved in Bramson, Ding and Zeitouni [15]. (Strictly speaking, [15] proves only the convergence in law for the value of the maximum in square-like domains. The above joint convergence can be gleaned from their proofs and/or can be found in Biskup and Louidor [10, 11], with [11] addressing general domains.) \square

We are now ready to deal with χ_k on $\Delta^k \setminus \Delta^{k-2}$ although, as mentioned before, not without some help from h'_k :

Lemma 3.10 *There are constants $c_1, c_2 \in (0, \infty)$ such that for all $k \geq 1$ and all $\lambda > 0$,*

$$P\left(\left|\max_{x \in \Delta^k \setminus \Delta^{k-1}} (\chi_k(x) + \chi_{k-1}(x) + h'_k(x)) - m_{2^k}\right| > \lambda\right) \leq c_1 e^{-c_2 \lambda}. \quad (3.43)$$

Similarly, we also have

$$P\left(\left|\max_{x \in \Delta^{k-1} \setminus \Delta^{k-2}} (\chi_k(x) + \chi_{k-1}(x) + \chi_{k-2}(x) + h'_{k-1}(x)) - m_{2^k}\right| > \lambda\right) \leq c_1 e^{-c_2 \lambda}. \quad (3.44)$$

Proof. Focusing first on the values in $\Delta^k \setminus \Delta^{k-1}$, the representation (3.15) shows that the difference between the field in the maximum and h^{Δ^k} equals

$$\begin{aligned} h^{\Delta^k}(x) - [\chi_k(x) + \chi_{k-1}(x) + h'_k(x)] \\ = (1 + \mathbf{b}_k(x)) \varphi_k(0) + (1 + \mathbf{b}_{k-1}(x)) \varphi_{k-1}(0). \end{aligned} \quad (3.45)$$

By Lemmas 3.6–3.7, the field on the right-hand side has uniform Gaussian tails which converts (3.43) to a bound on the tails of $\max_{x \in \Delta^k \setminus \Delta^{k-1}} h^{\Delta^k}(x) - m_{2^k}$. This bound is obtained by another reference to Lemma B.13, combined also with Lemma B.7 in the bound of the lower tail. The proof of (3.44) is completely analogous. \square

The last matter to address concerns a rewrite of the event $h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathbf{g}^{\Delta^n})$ in terms of the random objects $\varphi_k(0)$, χ_k and h'_k . For this we need to compare the quantity on the right to the natural growth of the maximum of h'_k which, as stated in Lemma 3.9, is captured by the sequence m_{2^k} . Here we note:

Lemma 3.11 *There is a constant $c \in (0, \infty)$ such that for all $n \geq 1$ and all $k = 0, \dots, n$,*

$$\max_{x \in \Delta^k \setminus \Delta^{k-1}} \left| m_{2^n} (1 - \mathfrak{g}^{\Delta^n}(x)) - m_{2^k} \right| \leq c + \frac{3}{4} \sqrt{g} \log(1 + k \wedge (n - k)). \quad (3.46)$$

Proof. We have

$$1 - \mathfrak{g}^{\Delta^n}(x) = \frac{G^{\Delta^n}(0, 0) - G^{\Delta^n}(0, x)}{G^{\Delta^n}(0, 0)}. \quad (3.47)$$

The Green function admits the representation (B.5) in Lemma B.3 which, using the asymptotic form of the potential from Lemma B.4, some straightforward calculations imply

$$1 - \mathfrak{g}^{\Delta^n}(x) = \frac{k}{n} + O\left(\frac{1}{n}\right), \quad x \in \Delta^k \setminus \Delta^{k-1}, \quad (3.48)$$

with the implicit constant in the error term uniform in k . Multiplying by m_{2^n} and invoking the explicit form (1.2) then yields the result. \square

3.4 Whole-space pinned field.

The estimates derived above permit us to make a connection to the pinned DGFF on all of \mathbb{Z}^2 and, in fact, realize this field on the same probability space as the objects φ_k , χ_k and h'_k :

Proposition 3.12 *Suppose $\{\varphi_k(0) : k \geq 0\}$, $\{\chi_k : k \geq 0\}$ and $\{h'_k : k \geq 0\}$ are independent objects with the laws as specified in Proposition 3.3 and let $\{\mathfrak{b}_k : k \geq 0\}$ be as in (3.14). Then for each $x \in \mathbb{Z}^2$, the infinite sum*

$$\phi(x) := \sum_{k \geq 0} \left(\mathfrak{b}_k(x) \varphi_k(0) + \chi_k(x) + h'_k(x) \right) \quad (3.49)$$

converges absolutely almost surely. Moreover, ϕ has the law ν^0 ; i.e., that of the (mean-zero) DGFF in $\mathbb{Z}^2 \setminus \{0\}$ or, the DGFF on \mathbb{Z}^2 pinned to zero at $x = 0$.

Proof. By standard Gaussian bounds and (3.28), $k \mapsto \varphi_k(0)$ grows at most polylogarithmically while, by Lemma 3.8, $k \mapsto \chi_k(x)$ decays exponentially. The value $h'_k(x)$ is (for a given x) non-zero only for one k . In light of (3.30), the sum in (3.49) thus converges almost surely.

In order to identify the law of the resulting field we observe (e.g., by comparison of covariances) that the DGFF in $\mathbb{Z}^2 \setminus \{0\}$ is the weak limit of $h^{\Delta^n} - h^{\Delta^n}(0)$ (as $n \rightarrow \infty$) which, by (3.4) and the fact that $\mathfrak{g}^{\Delta^n} \rightarrow 1$ pointwise, coincides with the weak limit of h^{Δ^n} conditioned on $h^{\Delta^n}(0) = 0$. Now (3.25) yields

$$h^{\Delta^n}(x) = \sum_{k=0}^n \left(\mathfrak{b}_k(x) \varphi_k(0) + \chi_k(x) + h'_k(x) \right) \quad \text{on } \{h^{\Delta^n}(0) = 0\}, \quad (3.50)$$

which is close to (3.49) except for one fact: conditioning on $h^{\Delta^n}(0) = 0$ changes the law of the variables $\{\varphi_k(0) : k = 0, \dots, n\}$.

To account for this change, note that, by (3.25) again and the Gaussian nature of all variables, conditional on $h^{\Delta^n}(0) = 0$, the law of $\{\varphi_k(0) : k = 0, \dots, n\}$ is that of

$$\left\{ \varphi_k(0) - c_n(k) \sum_{\ell=0}^n \varphi_\ell(0) : k = 0, \dots, n \right\} \quad (3.51)$$

under the unconditioned (product) measure, where

$$c_n(k) := \text{Var}(\varphi_k(0)) \left(\sum_{\ell=0}^n \text{Var}(\varphi_\ell(0)) \right)^{-1}. \quad (3.52)$$

Neither χ_k nor h'_k are affected by the conditioning, being independent of the $\varphi_k(0)$'s, and so

$$(h^{\Delta^n} | h^{\Delta^n}(0) = 0) \stackrel{\text{law}}{=} \tilde{\phi}_n, \quad (3.53)$$

where

$$\tilde{\phi}_n(x) := \sum_{k=0}^n \left(\mathbf{b}_k(x) \varphi_k(0) + \chi_k(x) + h'_k(x) \right) - \left(\sum_{k=0}^n \mathbf{b}_k(x) c_n(k) \right) \left(\sum_{\ell=0}^n \varphi_\ell(0) \right) \quad (3.54)$$

with all the variables now distributed as under the unconditioned measure.

Taking $n \rightarrow \infty$, the first term on the right of (3.54) tends to the infinite series (3.49) a.s. by our observations above. For the second term, by (3.28) and the absolute summability of the family of numbers $\{\mathbf{b}_k(x) : k \geq 0\}$, the first sum is of order n^{-1} . Since $\{\varphi_k(0) : k \geq 0\}$ are independent, Gaussian with mean zero and bounded variances, as $n \rightarrow \infty$ the whole second term in (3.54) tends to zero a.s. by the Law of Large Numbers. \square

Remark 3.13 Since the law ν^0 of the pinned DGFF ϕ is explicitly known (see (2.7–2.8)), we could have perhaps considered checking (3.49) directly by comparing covariances. Notwithstanding, we still find the above proof more illuminating; particularly, since we will re-use some of its arguments later.

4. REDUCTION TO A RANDOM WALK

The goal of this section is to rewrite the event $\{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n})\}$ from (3.3), as well as its \mathbb{Z}^2 -counterpart $\{\phi + \frac{2}{\sqrt{g}}\mathfrak{a} > 0 \text{ in } \Delta^n\}$, using the above random walk and well-behaved correction terms. The control of the resulting events for the random walk will require calculations for Brownian motion that are relegated to Appendix A. Although the random walk estimates are key for most subsequent derivations, as far as the main line of the proof is concerned, the main conclusions come in Lemmas 4.20–4.22 in Section 4.4.

4.1 Control variables.

We begin with the harder of the two events, $\{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n})\}$. The starting point is a definition of a random variable that will help us control the growth of the quantities $\varphi_k(0)$, χ_k and h'_k . To express the requisite errors, for $k, \ell = 1, \dots, n$ denote

$$\Theta_k(\ell) := \left[\log(1 + [k \vee (\ell \wedge (n - \ell))]) \right]^2. \quad (4.1)$$

Note that $k \mapsto \Theta_k(\ell)$ is increasing for each ℓ . The quantity also depends on n but we will keep that notationally suppressed. We then pose:

Definition 4.1 *Let K be the minimal $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ such that the following holds:*

- (1) *For each $\ell = 0, \dots, n$,*

$$|\varphi_\ell(0)| \leq \Theta_k(\ell), \quad (4.2)$$

(2) for each $\ell = 2, \dots, n$ and each $r = 0, \dots, \ell - 2$,

$$\max_{x \in \Delta^r} |\chi_\ell(x)| \leq 2^{(r-\ell)/2} \Theta_k(\ell), \quad (4.3)$$

(3) for each $\ell = 1, \dots, n$,

$$\left| \max_{x \in \Delta^\ell \setminus \Delta^{\ell-1}} (\chi_\ell(x) + \chi_{\ell-1}(x) + h'_\ell(x)) - m_{2^\ell} \right| \leq \Theta_k(\ell) \quad (4.4)$$

and

$$\left| \max_{x \in \Delta^\ell \setminus \Delta^{\ell-1}} (\chi_\ell(x) + \chi_{\ell-1}(x) + \chi_{\ell+1}(x) + h'_\ell(x)) - m_{2^\ell} \right| \leq \Theta_k(\ell). \quad (4.5)$$

If no such k exists, then we set $K := \lfloor n/2 \rfloor + 1$.

In the default case (i.e., $K := \lfloor n/2 \rfloor + 1$) no explicit bound on the above quantities can be assumed. However, this comes at little loss since we have:

Lemma 4.2 *There are $c > 0$ and $k_0 \geq 2$ such that*

$$P(K = k | S_{n+1} = 0) \leq e^{-c(\log k)^2}, \quad k = k_0, \dots, \lfloor n/2 \rfloor + 1. \quad (4.6)$$

In particular, for each $\delta \in (0, 1)$ and all n sufficiently large,

$$P(K > n^\delta | S_{n+1} = 0) \leq n^{-2}. \quad (4.7)$$

Proof. Recall (from the proof of Proposition 3.12) that, conditional on $S_{n+1} = 0$, the law of $\{\varphi_k(0) : k = 0, \dots, n\}$ is that of $\{\varphi_k(0) - c_n(k)S_{n+1} : k = 0, \dots, n\}$ under the unconditional measure, while χ_k and h'_k are not affected by the conditioning. By Lemmas 3.6 and 3.8, the numbers $\{nc_n(k) : k = 0, \dots, n\}$ are uniformly bounded in both k and n and the random variables

$$\{|\varphi_k(0) - c_n(k)S_{n+1}| : k = 0, \dots, n\} \quad \text{and} \quad \{2^{\ell-r} \max_{x \in \Delta^r} |\chi_\ell(x)| : 0 \leq r \leq \ell - 2, \ell \leq n\} \quad (4.8)$$

thus have uniform (in all indices involved) Gaussian tails. Lemma 3.9 in turn ensures that the random variables on the left of (4.5) have a uniform exponential tail.

Using the union bound, the probability that, in the conditional ensemble, condition (1) of Definition 4.1 fails is bounded by $2 \sum_{\ell \geq 0} e^{-c[\log(\ell \vee k)]^4}$, while the probability that (3) fails is bounded by a similar expression with 4 replaced by 2 in the exponent of the logarithm. The probability that condition (2) fails is in turn bounded by

$$\sum_{\ell=2}^{\infty} \sum_{r=0}^{\ell-2} e^{-c2^{\ell-r}[\log(\ell \vee k)]^4} \leq \sum_{i \geq 2} \sum_{\ell \geq 2} e^{-c2^i[\log(\ell \vee k)]^4}. \quad (4.9)$$

All of these error bounds combined yield no more than $e^{-c(\log k)^2}$ for a suitable $c > 0$ as soon as k is sufficiently large. This proves (4.6); the bound (4.7) is then immediate. \square

When the control variable is not defaulted to the maximal value, the above definitions ensure a rather tight control between the maximum of the field in the annuli $\Delta^k \setminus \Delta^{k-1}$ and a corresponding increment of the above random walk:

Lemma 4.3 (Approximation by a random walk) *There is a constant $C \in (0, \infty)$ such that if $K \leq \lfloor n/2 \rfloor$ holds for some $n \geq 1$, then for each $k = 0, \dots, n$ we have*

$$\left| \max_{x \in \Delta^k \setminus \Delta^{k-1}} [h^{\Delta^n}(x) - m_{2^n}(1 - \mathfrak{g}^{\Delta^n}(x))] - (S_{n+1} - S_k) \right| \leq R_K(k), \quad (4.10)$$

where $R_k(\ell) := C[1 + \Theta_k(\ell)]$.

Proof. Let $k = 1, \dots, n$ and pick $x \in \Delta^k \setminus \Delta^{k-1}$. Then (3.9), (3.29) and the last clause in Lemma 3.8 imply

$$\begin{aligned} h^{\Delta^n}(x) = S_{n+1} - S_k + \left(\sum_{\ell=k}^n \mathfrak{b}_\ell(x) \varphi_\ell(0) \right) + \left(\sum_{\ell=k+2}^n \chi_\ell(x) \right) \\ + \chi_{k-1}(x) + \chi_k(x) + \chi_{k+1}(x) + h'_k(x), \end{aligned} \quad (4.11)$$

where $\chi_{k+1}(x)$ is to be dropped when $k = n$. It follows from the definition of K and some elementary calculations that each of the two sums are bounded (in absolute value, uniformly in above x) by a quantity of the form $C[1 + \Theta_K(k)]$. An analogous bound holds also for the maximum of

$$\chi_{k-1}(x) + \chi_k(x) + \chi_{k+1}(x) + h'_k(x) - m_{2^k}, \quad (4.12)$$

again with χ_{k+1} is dropped when $k = n$. The claim follows by replacing $m_{2^n}(1 - \mathfrak{g}^{\Delta^n}(x))$ by m_{2^k} , which causes an error of the form $C[1 + \Theta_K(k)]$ as shown in Lemma 3.11. \square

The upshot of Lemma 4.3 is that the event $\{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n})\}$ can now be approximated by events defined solely in terms of the random walk $\{S_k : k = 1, \dots, n+1\}$ and the control variable K . Explicitly, we have:

$$\begin{aligned} \{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n})\} \cap \{K \leq \lfloor n/2 \rfloor\} \cap \{h^{\Delta^n}(0) = 0\} \\ \subseteq \{S_{n+1} = 0\} \cap \bigcap_{k=1}^n \{S_k \geq -R_K(k) - |t|\} \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n})\} \cap \{h^{\Delta^n}(0) = 0\} \\ \supseteq \{K \leq \lfloor n/2 \rfloor\} \cap \{S_{n+1} = 0\} \cap \bigcap_{k=1}^n \{S_k \geq R_K(k) + |t|\}, \end{aligned} \quad (4.14)$$

where we also recalled that $0 \leq 1 - \mathfrak{g}^{\Delta^n} \leq 1$.

Our control of the event $\{\phi + \frac{2}{\sqrt{g}}\mathfrak{a} > 0 \text{ in } \Delta^n\}$ will be quite similar, and will use only slightly modified versions of the above concepts. For $k, \ell \geq 1$, let

$$\tilde{\Theta}_k(\ell) := [\log(1 + k \vee \ell)]^2, \quad (4.15)$$

which we can view as the $n \rightarrow \infty$ limit of $\Theta_k(\ell)$ above. For the analogue of the control variable K , we put forward the following definition:

Definition 4.4 *Let \tilde{K} be the smallest $k \geq 2$ such that, for all $n \geq 2$, the bounds in Definition 4.1 hold with Θ_k replaced by $\tilde{\Theta}_k$. (If no such k exists, we set $\tilde{K} := \infty$.)*

The tails of the control variable \tilde{K} are easy to bound:

Lemma 4.5 *There is $c > 0$ and $k_0 \geq 2$ such that*

$$P(\tilde{K} \geq k) \leq e^{-c(\log k)^2}, \quad k \geq k_0. \quad (4.16)$$

In particular, $\tilde{K} < \infty$ almost surely.

Proof. The proof is nearly identical to that of Lemma 4.2; in fact, it is much easier due to the absence of conditioning on $S_{n+1} = 0$. \square

The setup in Definition 4.4 then again implies:

Lemma 4.6 *Let ϕ be as in (3.49). Then there exists a constant $C > 0$ such that, on $\{\tilde{K} < \infty\}$,*

$$\left| \min_{x \in \Delta^\ell \setminus \Delta^{\ell-1}} \left(\phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) \right) - S_\ell \right| \leq \tilde{R}_{\tilde{K}}(\ell), \quad \ell \geq 1, \quad (4.17)$$

where $\tilde{R}_k(\ell) := C[1 + \tilde{\Theta}_k(\ell)]$.

Proof. Let $\ell \geq 1$ and pick $x \in \Delta^\ell \setminus \Delta^{\ell-1}$. Then (3.29) and the various properties of the random objects $\phi_j(0)$, χ_j and h'_j imply

$$\begin{aligned} \phi(x) = -S_\ell + \sum_{j \geq \ell} \mathbf{b}_j(x) \phi_j(0) \\ + \sum_{j \geq \ell+2} \chi_j(x) + [\chi_{\ell-1}(x) + \chi_\ell(x) + \chi_{\ell+1}(x) + h'_\ell(x)]. \end{aligned} \quad (4.18)$$

As in the proof of Lemma 4.3, the first two sums are bounded (in absolute value) by a quantity of the form $C[1 + \Theta_k(\ell)]$ for some constant $C > 0$, uniformly in above x . Similarly, we get

$$\left| \min_{x \in \Delta^\ell \setminus \Delta^{\ell-1}} [\chi_{\ell-1}(x) + \chi_\ell(x) + \chi_{\ell+1}(x) + h'_\ell(x)] - m_{2^\ell} \right| \leq C[1 + \Theta_k(\ell)]. \quad (4.19)$$

The claim now follows from

$$\max_{x \in \Delta^\ell} \left| \frac{2}{\sqrt{g}} \mathbf{a}(x) - m_{2^\ell} \right| \leq C[1 + \Theta_k(\ell)], \quad (4.20)$$

as implied by (1.2) and the asymptotic form of \mathbf{a} (see Lemma B.4). \square

As a consequence of the above observations, we again get a tight control between the event in Theorems 2.3 and 2.4 and the above random walk:

$$\{\tilde{K} < \infty\} \cap \bigcap_{\ell=1}^n \{S_\ell \geq \tilde{R}_{\tilde{K}}(\ell)\} \subseteq \left\{ \phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) > 0 : x \in \Delta^n \right\} \quad (4.21)$$

and

$$\{\tilde{K} < \infty\} \cap \left\{ \phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) > 0 : x \in \Delta^n \right\} \subseteq \{\tilde{K} < \infty\} \cap \bigcap_{\ell=1}^n \{S_\ell \geq -\tilde{R}_{\tilde{K}}(\ell)\} \quad (4.22)$$

The upshot of (4.13–4.14) and (4.21–4.22) is that the two events $\{h^{\Delta^n} \leq (m_{2^n} + t)(1 - \mathbf{g}^{\Delta^n})\}$ and \mathbb{Z}^2 -counterpart $\{\phi + \frac{2}{\sqrt{g}} \mathbf{a} > 0 \text{ in } \Delta^n\}$ of our prime interest can be represented, via bounds, as events that the random walk $\{S_k : k = 1, \dots, n+1\}$ stays above a polylogarithmic curve. We thus need to find a way to efficiently control the probability of such random-walk events.

4.2 Brownian motion above a curve.

It is well-known that the simple random walk bridge (from zero to zero) of time-length n stays positive with probability that decays proportionally to n^{-1} . An elegant proof exists, based on a symmetry argument, which applies to rather general walks with time-homogeneous steps. Unfortunately, our problem is harder for the following reasons:

- (1) the steps of our random walk $\{S_k: k = 0, \dots, n\}$ are only approximately time-homogeneous (see (3.28) for a precise statement),
- (2) the events in (4.13–4.14) only compare the random walk to polylogarithmic curves and so an argument based solely on symmetry is not possible,
- (3) the asymptotic probability of the giant intersections in (4.13) and (4.14) will differ by a multiplicative constant, due to a difference in the restriction near the endpoints.

We thus have to develop tools to address these differences. Based on experience gained in the context of Branching Brownian Motion by Bramson [14], it is easier to first deal with corresponding claims for Brownian motion and Brownian bridge.

Let $\{B_t: t \geq 0\}$ be the standard Brownian motion and let P^x be the law with $P^x(B_0 = x) = 1$. We will represent the “curve” by a function $\zeta: [0, \infty) \rightarrow [0, \infty)$. We will separately deal with the cases of curves that are positive (i.e., the case when we require $B \geq \zeta$) or negative (i.e., the case of $B \geq -\zeta$). We begin with (unconditioned) Brownian motion above a positive curve:

Proposition 4.7 *For $\zeta: [0, \infty) \rightarrow [0, \infty)$ non-decreasing, continuous and such that $\zeta(s) = o(s^{1/2})$ as $s \rightarrow \infty$, let*

$$\rho(x) := \zeta(x^2) + \frac{x}{2} \int_{x^2}^{\infty} \frac{\zeta(s)}{s^{3/2}} ds. \quad (4.23)$$

Then for all $t > 0$ and all $x > \zeta(0)$,

$$P^x(B_s \geq \zeta(s): s \in [0, t]) \geq (1 - \delta) \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}} \quad (4.24)$$

holds with

$$\delta := \frac{x^2}{2t} + 4 \left(\frac{\rho(x)}{x} \right)^{2/3}. \quad (4.25)$$

Clearly, the bound is not useful unless $x^2 \ll t$ and $\rho(x) \ll x$. It turns out that roughly the same conditions guarantee good control for Brownian bridge above a positive curve as well:

Proposition 4.8 *Let ζ and ρ be as in Proposition 4.7. Then for each $t > 0$ and all $x, y > \zeta(0)$,*

$$P^x(B_s \geq \zeta(s \wedge (t-s)): s \in [0, t] \mid B_t = y) \geq (1 - \delta) \frac{2xy}{t} \quad (4.26)$$

holds with

$$\delta := \frac{xy}{t} + 4 \left(\sqrt{\frac{\rho(x)}{x}} + \sqrt{\frac{\rho(y)}{y}} \right) e^{\frac{(x-y)^2}{2t}}. \quad (4.27)$$

The proofs of Propositions 4.7 and 4.8 are long and technical and would detract from the main line of presentation. Hence we defer them to Appendix A.

Next we move to the case of negative curves. We again start with the case of unconditioned Brownian motion:

Proposition 4.9 *For $\zeta: [0, \infty) \rightarrow [0, \infty)$ non-decreasing and continuous and with $\zeta(s) = o(s^{1/4})$ as $s \rightarrow \infty$, denote*

$$\tilde{\rho}(x) := \rho(x) + 4 \frac{\zeta(x^2)^2}{x} + 2 \int_{x^2}^{\infty} \frac{\zeta(s)^2}{s^{3/2}} ds \quad (4.28)$$

where $\rho(x)$ is as in (4.23). Then for all $t > 0$ and all $x > 0$,

$$P^x\left(B_s \geq -\zeta(s) : s \in [0, t]\right) \leq (1 + \delta) \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}} \quad (4.29)$$

holds true with $\delta := \kappa_1\left(\frac{\tilde{\rho}(x)}{x}\right)$ for $\kappa_1(u) := 4(1 + u^{2/3})u^{2/3}$.

For the bound to be useful, we need that $\delta \ll 1$ which in turn requires that $\tilde{\rho}(x) \ll x$. Not too surprisingly, the same criterion also applies to the case of the Brownian bridge:

Proposition 4.10 *Suppose ζ and $\tilde{\rho}$ are as in Proposition 4.7. Then for all $t > 0$ and all $x, y > 0$,*

$$P^x\left(B_s \geq -\zeta(s \wedge (t-s)) : s \in [0, t] \mid B_t = y\right) \leq (1 + \delta) \frac{2xy}{t} \quad (4.30)$$

holds true with $\delta := \kappa_2\left(\frac{\tilde{\rho}(x)}{x}, \frac{\tilde{\rho}(y)}{y}\right) e^{\frac{(x-y)^2}{2t}}$ for $\kappa_2(u, v) := 48(1+u)(1+v)(\sqrt{u} + \sqrt{v})$.

The proofs of Propositions 4.9 and 4.10 are again relegated to Appendix A. The reader should note that, despite some similarities between the cases of positive and negative curves, there are also notable differences. Indeed, in order to have $\rho(x) \ll x$ it suffices to have $\zeta(s) = o(s^{1/2})$, while for $\tilde{\rho}(x) \ll x$, we seem to need $\zeta(s) = o(s^{1/4})$. While the former criterion is basically best possible, the latter is likely not optimal. However, as the above statements are fully sufficient for our needs, we have not tried to bring them to a necessarily optimal form.

Remark 4.11 We note that, in his groundbreaking study of the Branching Brownian Motion, Bramson [14, Propositions 1, 1', 2 and 2'] proved bounds of the form (4.24), (4.26), (4.29) and (4.30) for the specific choice $\zeta(s) := (3/\sqrt{8})\log(s \vee 1)$ and $x = y$. Unfortunately, we are not able to use his conclusions for two reasons: First, Bramson's upper and lower bounds differ by an overall multiplicative constant which is something that our applications of these bounds cannot tolerate. Second, we need to control a whole class of ζ 's uniformly.

In order to use the above statements in various situations of interest, it will be convenient to have a quick tool for bounding the quantity $\tilde{\rho}(x)$ (which automatically bounds also $\rho(x)$) for a reasonably large class of ζ . This is the subject of:

Lemma 4.12 *Let $\zeta : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing, continuously differentiable and such that $\zeta(0) > 0$ and, for some $a, \sigma > 0$,*

$$\zeta'(s) \leq a \frac{\log(1 + s/\sigma^2)}{1 + s/\sigma^2}, \quad s > 0. \quad (4.31)$$

For $u \geq 0$ set $\zeta_u(s) := \zeta(u + s)$ and let $\tilde{\rho}_u(x)$ be the quantity in (4.28) associated with ζ_u . There is a constant $c = c(a, \sigma)$ such that for all $x \geq 1$ and all $u \geq 0$,

$$\tilde{\rho}_u(x) \leq 2\zeta(u) + 16 \frac{\zeta(u)^2}{x} + c \left(\log\left(e + \frac{x^2}{\sigma^2}\right) \right)^4 \quad (4.32)$$

Proof. Since $\frac{\log(1+s)}{1+s} \leq e \frac{\log(e+s)}{e+s}$, with the expression on the right now decreasing on $\{s \geq 0\}$, integrating the inequality (4.31) yields

$$\zeta_u(s) \leq \zeta(u) + \frac{1}{2} a \sigma^2 e \left[1 + \log\left(e + \frac{s}{\sigma^2}\right) \right]^2. \quad (4.33)$$

To get the result, plug this in the expression for $\tilde{\rho}$, use $(a+b)^2 \leq 2a^2 + 2b^2$ to deal with the quadratic occurrences of $\zeta(s)$ and perform a sequence of integrations by parts. \square

The above calculations permit us to make a statement concerning *entropic repulsion*, which amounts to the fact that, conditioning a Brownian path to stay above a negative curve, it will stay above a positive curve except perhaps near the starting point.

Proposition 4.13 *Let $\zeta: [0, \infty) \rightarrow [0, \infty)$ depend on a and σ as in Lemma 4.12. There are constants $c = c(a, \sigma) > 0$ and $c' = c'(a, \sigma, \zeta(0)) > 0$ such that for all $t > 2c'$ and all $u \in [c', t/2]$,*

$$P^0\left(\min_{0 \leq s \leq t} [B_s + \zeta(s)] > 0 > \min_{u \leq s \leq t} [B_s - \zeta(s)]\right) \leq cu^{-\frac{1}{16}} \frac{1}{\sqrt{t}}. \quad (4.34)$$

This statement is far from optimal — in fact, one expects that the Brownian motion will stay above the curve $s \mapsto s^\delta$ for each $\delta < 1/2$ — but the above is easy to prove given what we already have and is completely sufficient for our needs. The proof is given in Section A.5. A similar statement holds for the Brownian bridge as well:

Proposition 4.14 *Let ζ be as in Proposition 4.13 and abbreviate $\tilde{\zeta}(s) := \zeta(s \wedge (t-s))$. Then there are constants $\tilde{c} = \tilde{c}(a, \sigma) > 0$ and $\tilde{c}' = \tilde{c}'(a, \sigma, \zeta(0)) > 0$ such that for all sufficiently large $t > 0$ and all $u \in [\tilde{c}', t/4]$,*

$$P^0\left(\min_{0 \leq s \leq t} [B_s + \tilde{\zeta}(s)] > 0 > \min_{u \leq s \leq t} [B_s - \tilde{\zeta}(s)] \mid B_t = 0\right) \leq \tilde{c}u^{-\frac{1}{16}} \frac{1}{t}. \quad (4.35)$$

The proof proceeds by a reduction to Proposition 4.13 and is therefore also deferred to Section A.5. As before, we believe that the path gets repelled above a curve $s \mapsto (s \wedge (t-s))^\delta$ for every $\delta < 1/2$, except perhaps near the endpoints.

4.3 Random walk above a curve.

Our next task is to convert the above statements for Brownian motion to statements about random walks. We will use the convenient fact that our random walks have mean-zero Gaussian steps and so we may as well realize them as values of a standard Brownian motion observed at a deterministic sequence of times. Let $\{\sigma_k^2: k \geq 0\}$ denote a sequence of numbers obeying

$$0 < \inf_{k \geq 0} \sigma_k^2 \leq \sup_{k \geq 0} \sigma_k^2 < \infty. \quad (4.36)$$

Define

$$t_k := \sum_{\ell=0}^{k-1} \sigma_\ell^2, \quad k \geq 1, \quad (4.37)$$

with $t_0 := 0$. Note that, for the specific choice $\sigma_k^2 := \text{Var}(\varphi_k(0))$ — not necessarily assumed in the discussion later — we get

$$\{S_k: k = 1, \dots, n+1\} \stackrel{\text{law}}{=} \{B_{t_k}: k = 1, \dots, n+1\}. \quad (4.38)$$

The reduction of the key statements from Brownian motion to the random walk will be considerably simplified using the following claim:

Lemma 4.15 *For $\{\sigma_k^2: k \geq 0\}$ and $\{t_k: k \geq 0\}$ as above, let σ_{\min}^2 , resp., σ_{\max}^2 denote the infimum, resp., the supremum in (4.36). Given integers $n \geq 1$ and $k \leq \lfloor n/2 \rfloor$ and a non-decreasing*

concave function $\gamma: [0, \infty) \rightarrow [0, \infty)$, let

$$\zeta(s) := \gamma\left(\frac{t_k + s}{\sigma_{\min}^2}\right). \quad (4.39)$$

Then for all $x, y \in \mathbb{R}$,

$$\begin{aligned} P^0\left(B_{t_\ell} \geq -\gamma(\ell \wedge (n - \ell)) : \ell = k, \dots, n - k \mid B_{t_k} = x, B_{t_{n-k}} = y\right) \\ \leq P^x\left(B_s \geq -2\zeta(s \wedge (t_{n-k} - t_k - s)) : s \in [0, t_{n-k} - t_k] \mid B_{t_{n-k} - t_k} = y\right) \\ \times \prod_{j=k}^{n-k} \left(1 - e^{-2\sigma_{\max}^{-2}\gamma(j)^2}\right)^{-2}. \end{aligned} \quad (4.40)$$

Similarly, if $\tilde{\zeta}$ is defined by

$$\tilde{\zeta}(s) := \gamma\left(\frac{t_k + s}{\sigma_{\max}^2}\right), \quad (4.41)$$

then for all $x, y \in \mathbb{R}$,

$$\begin{aligned} P^0\left(B_{t_\ell} \geq -\gamma(\ell \wedge (n - \ell)) : \ell = k, \dots, n - k \mid B_{t_k} = x, B_{t_{n-k}} = y\right) \\ \geq P^x\left(B_s \geq -\tilde{\zeta}(s \wedge (t_{n-k} - t_k - s)) : s \in [0, t_{n-k} - t_k] \mid B_{t_{n-k} - t_k} = y\right). \end{aligned} \quad (4.42)$$

Notice that the function $s \mapsto \zeta(s \wedge (t_{n-k} - t_k - s))$ is symmetric about the midpoint of the interval $[0, t_{n-k} - t_k]$. This will be important as soon as we try to apply the earlier conclusions to the probabilities on the right-hand sides of (4.40) and (4.42).

The proof of Lemma 4.15 will rely on the following bounds:

Lemma 4.16 *Given sequences $\{\sigma_k^2 : k \geq 1\}$ and $\{t_k : k \geq 0\}$ as above, let $\zeta : [0, t_n] \rightarrow [0, \infty)$ be a non-decreasing concave function. Then for each $x, y \geq 0$,*

$$\begin{aligned} P^x(B_s \geq -2\zeta(s) : 0 \leq s \leq t_n \mid B_{t_n} = y) & \prod_{k=1}^n \left(1 - e^{-2\sigma_k^{-2}\zeta(t_{k-1})^2}\right)^{-1} \\ & \geq P^x(B_{t_k} \geq -\zeta(t_k) : k = 0, \dots, n \mid B_{t_n} = y) \\ & \geq P^x(B_s \geq -\zeta(s) : 0 \leq s \leq t_n \mid B_{t_n} = y). \end{aligned} \quad (4.43)$$

Proof. The inequality on the right is trivial so let us focus on that on the left. Let

$$W^{(k)}(s) := \frac{t_{k+1} - s}{t_{k+1} - t_k} B_{t_k} + \frac{s - t_k}{t_{k+1} - t_k} B_{t_{k+1}} - B_s, \quad t_k \leq s \leq t_{k+1}. \quad (4.44)$$

Under the conditional measure $P^x(-\mid B_{t_n} = y)$, the processes $\{W^{(k)} : k \geq 0\}$ have the law of a family of independent Brownian bridges — from zero to zero, with $W^{(k)}$ indexed by times in the interval $[t_k, t_{k+1}]$ — and this family is independent of the values $\{B_{t_k} : k = 0, \dots, n\}$. Since ζ is concave and non-decreasing on the intervals $[t_k, t_{k+1}]$, we have

$$\begin{aligned} \{B_s \geq -2\zeta(s) : 0 \leq s \leq t_n\} \\ \supseteq \{B_{t_k} \geq -\zeta(t_k) : k = 0, \dots, n\} \cap \bigcap_{k=0}^{n-1} \left\{ \max_{t_k \leq s \leq t_{k+1}} W^{(k)}(s) \leq \zeta(t_k) \right\}. \end{aligned} \quad (4.45)$$

By the Reflection Principle (cf (A.7)), the probability of the event in the giant intersection corresponding to index k is equal to $1 - e^{-2\sigma_{k+1}^{-2}\zeta(t_k)^2}$. Using the stated independence, we now readily get the left inequality in (4.43) as well. \square

Proof of Lemma 4.15. Abbreviate $\widehat{\zeta}(s) := \zeta(s \wedge (t_{n-k} - t_k - s))$. Noting that

$$t_\ell \geq \sigma_{\min}^2 \ell \quad \text{and} \quad t_{n-k} + t_k - t_\ell \geq \sigma_{\min}^2 (n - \ell), \quad \ell = k, \dots, n - k, \quad (4.46)$$

we have

$$\widehat{\zeta}(t_\ell - t_k) \geq \gamma\left(\frac{1}{\sigma_{\min}^2}(t_\ell \wedge (t_{n-k} + t_k - t_\ell))\right) \geq \gamma(\ell \wedge (n - \ell)) \quad (4.47)$$

for all $\ell = k, \dots, n - k$. This yields

$$\begin{aligned} & P^0\left(B_{t_\ell} \geq -\gamma(\ell \wedge (n - \ell)) : \ell = k, \dots, n - k \mid B_{t_k} = x, B_{t_{n-k}} = y\right) \\ & \leq P^x\left(B_{t_\ell - t_k} \geq -\widehat{\zeta}(t_\ell - t - t_k) : \ell = k, \dots, n - k \mid B_{t_{n-k} - t_k} = y\right) \end{aligned} \quad (4.48)$$

We now use (4.43) to estimate this by

$$P^x\left(B_s \geq -2\widehat{\zeta}(s) : s \in [0, t_{n-k} - t_k] \mid B_{t_{n-k} - t_k} = y\right) \prod_{\ell=k+1}^{n-k} \left(1 - e^{-2\sigma_\ell^{-2}\widehat{\zeta}(t_{\ell-1})^2}\right)^{-1}. \quad (4.49)$$

From (4.47) it is now easy to check that the product is bounded by that in (4.40). This proves the upper bound. For the lower bound (4.42) we replace ζ by $\tilde{\zeta}$ in the definition of $\widehat{\zeta}$ and notice that the opposite inequality then holds in (4.47). Lemma 4.16 then readily yields the claim. \square

We will also need a similar statement for the unconditioned random walk. (Unfortunately, due to the symmetrization of the function ζ in (4.40) and (4.42), this does not follow by mere integration over y .) Naturally, the expressions are considerably simpler in this case:

Lemma 4.17 *Let γ be a function as in Lemma 4.15. For each $x \in \mathbb{R}$, each $n \geq k \geq 1$ and ζ defined from γ and k as in (4.39),*

$$\begin{aligned} & P^0\left(B_{t_\ell} \geq -\gamma(\ell) : \ell = k, \dots, n \mid B_{t_k} = x\right) \\ & \leq P^x\left(B_s \geq -2\zeta(s) : s \in [0, t_n - t_k]\right) \prod_{j=k}^n \left(1 - e^{-2\sigma_{\max}^{-2}\gamma(j)^2}\right)^{-2}. \end{aligned} \quad (4.50)$$

Similarly, for all $x \in \mathbb{R}$ and all $n \geq k \geq 1$ and $\tilde{\zeta}$ as in (4.41),

$$P^0\left(B_{t_\ell} \geq -\gamma(\ell) : \ell = k, \dots, n \mid B_{t_k} = x\right) \geq P^x\left(B_s \geq -\tilde{\zeta}(s) : s \in [0, t_n - t_k]\right). \quad (4.51)$$

Proof. The proof follows exactly the same lines as that of Lemma 4.15 and is in fact easier due to the absence of symmetrization. Hence, it is omitted. \square

We will now move to the statement of entropic repulsion for the above random walk. We will henceforth work with

$$\gamma(s) := a[1 + \log(a + s)]^2 \quad \text{where} \quad a > \frac{1 + \sqrt{5}}{2}. \quad (4.52)$$

As a calculation shows, the restriction on a ensures that γ is non-decreasing and concave on $[0, \infty)$. For a change, we start with the claim for the unconditioned random walk:

Lemma 4.18 *Let γ be as in (4.52). Then there is $c = c(a) \in (0, \infty)$ such that for all sufficiently large $n \geq 1$ and all $k \geq 1$ with $k \leq \frac{n}{2}$,*

$$\frac{P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \gamma(\ell)] > 0 > \min_{k \leq \ell \leq n} [B_{t_\ell} - \gamma(\ell)]\right)}{P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \gamma(\ell)] > 0\right)} \leq ck^{-\frac{1}{16}}. \quad (4.53)$$

Proof. Let $\zeta^{\min}(s) := \gamma(s/\sigma_{\min}^2)$. Then $\zeta^{\min}(t_\ell) \geq \gamma(\ell)$ and (4.45) thus implies

$$\begin{aligned} & P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \gamma(\ell)] > 0 > \min_{k \leq \ell \leq n} [B_{t_\ell} - \gamma(\ell)]\right) \prod_{j=1}^n \left(1 - e^{-2\sigma_{\max}^{-2}\gamma(j)^2}\right) \\ & \leq P^0\left(\min_{0 \leq s \leq t_n} [B_s + \zeta^{\min}(s)] > 0 > \min_{t_k \leq s \leq t_n} [B_s - \zeta^{\min}(s)]\right) \end{aligned} \quad (4.54)$$

The product is bounded away from zero uniformly in n and so, Proposition 4.13 and the fact that $t_k \geq \sigma_{\min}^2 k$, the right-hand side is at most $ck^{-\frac{1}{16}}/\sqrt{t_n}$ as soon as k is large enough.

For a suitable lower bound on the denominator, here we set $\zeta^{\max}(s) := \gamma(s/\sigma_{\max}^2)$ and note that $\gamma(\ell) \geq \zeta^{\max}(\ell)$. By Proposition 4.13,

$$\begin{aligned} & P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \gamma(\ell)] > 0\right) \geq P^0\left(\min_{0 \leq s \leq t_n} [B_s + \zeta^{\max}(s)] > 0\right) \\ & \geq P^0\left(\min_{u \leq s \leq t_n} [B_s - \zeta^{\max}(s)] > 0\right) - cu^{-\frac{1}{16}} \frac{1}{\sqrt{t_n}}, \end{aligned} \quad (4.55)$$

where $c = c(a, \sigma)$ and $u \geq c' = c'(a, \sigma, \zeta(0))$. By Lemma 4.12 for ζ^{\max} in place of ζ , the quantity $\sup_{x \geq u} \tilde{\rho}_u(x)/x$ is bounded and tending to zero with $u \rightarrow \infty$. Hence, for u large enough, Proposition 4.7 bounds the probability on the right of (4.55) by $\tilde{c}/\sqrt{t_n}$ with some \tilde{c} independent of $(a, \sigma, \zeta(0))$. The claim follows. \square

Lemma 4.19 *Let γ be as in (4.52) and set $\tilde{\gamma}(k) := \gamma(k \wedge (n-k))$. There is $c = c(a) \in (0, \infty)$ such that for all sufficiently large $n \geq 1$ and all $k \geq 1$ with $k \leq \frac{n}{4}$,*

$$\frac{P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \tilde{\gamma}(\ell)] > 0 > \inf_{k \leq \ell \leq n-k} [B_{t_\ell} - \tilde{\gamma}(\ell)] \mid B_{t_n} = 0\right)}{P^0\left(\min_{0 \leq \ell \leq n} [B_{t_\ell} + \tilde{\gamma}(\ell)] > 0 \mid B_{t_n} = 0\right)} \leq ck^{-\frac{1}{16}}. \quad (4.56)$$

Proof. The proof is completely analogous to that of Lemma 4.18; we just use Lemma 4.15 instead of Lemma 4.17 and Proposition 4.14 instead of Proposition 4.13. \square

4.4 Growth and gap control.

Returning back to our consideration of the pinned DGFF, we are now in a position to derive quantitative estimates on various undesirable events of interest. We start by bounding the probability that the control variables K , resp., \tilde{K} introduced earlier take a given value:

Lemma 4.20 *There are constants $c_1, c_2 \in (0, \infty)$ such that for all $n \geq 1$ sufficiently large, all $k \geq 1$ with $k < n/4$ and all $t \geq 0$,*

$$P\left(\{K = k\} \cap \bigcap_{\ell=k}^{n-k} \{S_\ell \geq -R_k(\ell) - t\} \mid S_{n+1} = 0\right) \leq c_1(1+t)^2 \frac{e^{-c_2(\log k)^2}}{n}. \quad (4.57)$$

Similarly, there are constants $\tilde{c}_1, \tilde{c}_2 \in (0, \infty)$ such that for all $r \geq k \geq 1$,

$$\nu^0\left(\{\tilde{K} = k\} \cap \bigcap_{\ell=k}^r \{S_\ell \geq -\tilde{R}_k(\ell)\}\right) \leq \tilde{c}_1 \frac{e^{-\tilde{c}_2(\log k)^2}}{\sqrt{r}}. \quad (4.58)$$

Proof. Given $k \geq 1$, let A_k denote the event that the conditions in Definition 4.1 (with k as stated) hold for all ℓ satisfying $\ell \leq k$ or $\ell \geq n - k$, but at least one of these conditions fails when k is replaced by $k - 1$. Note that A_k belongs to the σ -algebra

$$\mathcal{F}_k := \sigma\left(\varphi_\ell(0), \chi_\ell, h'_\ell : \ell = 0, \dots, k, n - k, \dots, n\right) \quad (4.59)$$

and that $\{K = k\} \subseteq A_k$. The desired probability is thus bounded by

$$E\left(1_{A_k} P\left(\bigcap_{\ell=k}^{n-k} \{S_\ell \geq -R_k(\ell) - t\} \mid \mathcal{F}_k\right) \mid S_{n+1} = 0\right). \quad (4.60)$$

Our strategy is to derive a pointwise estimate on the conditional probability.

First we note that, setting γ to (4.52) with $a > \max\{C, \frac{1+\sqrt{5}}{2}\}$ where C is the constant in Lemma 4.3, on the event $\{S_k = x, S_{n-k} = y\}$ the above conditional probability can be bounded above by

$$P^0\left(B_{t_\ell} \geq -\gamma(\ell \wedge (n - \ell)) - t : \ell = k, \dots, n - k \mid B_{t_k} = x, B_{t_{n-k}} = y\right), \quad (4.61)$$

where B is the standard Brownian motion and $\{t_k : k \geq 1\}$ — not to be confused with t and t_0 in the statement — are now defined using $\sigma_k^2 := \text{Var}(\varphi_k(0))$. Setting $\zeta(s) := \gamma(s/\sigma_{\min}^2)$ and comparing this with (4.39), we bound this probability using (4.40) with ζ replaced by $\zeta_u(s) := \zeta(u + s)$ where $u := t_k$. Here we observe that, for our choice of γ , the product on the right of (4.40) is bounded by a constant uniformly in k, n and $t \geq 0$. Using a trivial shift of coordinates, we thus need to derive a uniform bound on

$$P^{x+t}\left(B_s \geq -2\zeta_{t_k}(s \wedge (t_{n-k} - t_k - s)) : s \in [0, t_{n-k} - t_k] \mid B_{t_{n-k}-t_k} = y + t\right) \quad (4.62)$$

for all relevant x and y . For this we note that, on A_k , we necessarily have $|\varphi_\ell(0)| \leq \Theta_k(k)$ for all $\ell \leq k$ and $\ell \geq n - k$ and so, since we condition on $S_{n+1} = 0$, we may assume that

$$-\gamma(k \wedge (n - k)) \leq S_k, S_{n-k} \leq (k + 1)\Theta_k(k) = (k + 1)[\log(k + 1)]^2. \quad (4.63)$$

Since (4.62) increases when both x and y are increased by the same amount, by adding to x and y two times $a_k := \gamma(k) \vee ((k + 1)[\log(k + 1)]^2)$ it thus suffices to estimate the maximal value the probability in (4.62) takes for $a_k \leq x, y \leq 3a_k$.

We will use Proposition 4.10 but for that we first need to bound the error term constituting δ . Since we work with ζ_u , we first check that (4.31) in Lemma 4.12 applies with a as above and $\sigma^2 := \sigma_{\min}^2$, and so the requisite $\tilde{\rho}_u$ for $u := t_k$ is can be estimated as in (4.32). Hence we get

$$\sup_{t \geq 0} \max_{k=1, \dots, n} \sup_{a_k \leq x \leq 3a_k} \frac{\tilde{\rho}_{t_k}(x + t)}{x + t} < \infty. \quad (4.64)$$

As $t_{n-k} - t_k \geq c'n$ for some $c' > 0$ due to our assumption that $k < n/4$, (4.30) in conjunction with the previous steps yield

$$P\left(\bigcap_{\ell=k}^{n-k} \{S_\ell \geq -R_k(\ell) - t\} \mid \mathcal{F}_k\right) \leq c_1 \frac{(3a_k + t)^2}{n} \quad \text{on } A_k \cap \{S_{n+1} = 0\}, \quad (4.65)$$

for some constant $c_1 > 0$. The expectation in (4.60) is then at most $c_1 \frac{(3a_k + t)^2}{n} P(A_k \mid S_{n+1} = 0)$ and the last probability is estimated as in Lemma 4.2 by $e^{-c_2(\log k)^2}$. Sacrificing part of the exponent, we can now rewrite this as (4.57).

The proof of the corresponding statement (4.58) for Brownian bridge follows exactly the same argument as that of (4.57); in fact, the derivation is simpler due to the absence of n and t dependence of all terms. We leave the details to the reader. \square

As a consequence of these bounds, we are now able to control the leading order of the probability of the principal events of interest:

Lemma 4.21 *There are a constant $c_2 \in (0, \infty)$ and for each $t_0 > 0$ also a constant $c_1 \in (0, \infty)$ such that for all $n \geq 1$ and all $s \in [0, t_0]$ and all $t \leq s$, we have*

$$\frac{c_1}{n} \leq P\left(h^{\Delta^n} \leq m_{2^n} + s - (m_{2^n} + t)g^{\Delta^n} \mid h^{\Delta^n}(0) = 0\right) \leq \frac{c_2}{n}(1 + s - t)^2. \quad (4.66)$$

Similarly, there are $c'_1, c'_2 \in (0, \infty)$ such that for all $r \geq 1$,

$$\frac{c'_1}{\sqrt{r}} \leq \nu^0\left(\phi(x) + \frac{2}{\sqrt{g}}a(x) \geq 0 : x \in \Delta^r\right) \leq \frac{c'_2}{\sqrt{r}}. \quad (4.67)$$

Proof. Let us start with the upper bound in (4.66). For the event under consideration we get

$$\begin{aligned} \{h^{\Delta^n} - m_{2^n}(1 - g^{\Delta^n}) \leq (s - t)g^{\Delta^n}\} \\ \subseteq \{K = \lfloor n/2 \rfloor + 1\} \cup \bigcap_{k=1}^n \left(\{S_k - S_{n+1} \geq -R_k(k) - (s - t)\} \cap \{K \leq \lfloor n/2 \rfloor\}\right). \end{aligned} \quad (4.68)$$

Here we bounded $(s - t)g^{\Delta^n} \leq (s - t)$ and then invoked Lemma 4.3. In light of (3.25), the upper bound in (4.66) now follows by summing (4.57) over $1 \leq k \leq n/4$ and applying (4.7) to deal with the complementary values of the control variable. (The contribution of the latter part is then absorbed into that of the former.)

The upper bound in (4.67) is completely analogous; we invoke Lemma 4.6 to rewrite the event using the random walk and the control variable and then bound the resulting probability by (4.58) for $k \leq r$ and (4.16) for $k \geq r$ (including the default value $\tilde{K} = \infty$).

For the lower bounds, we will conveniently use the fact that the law of both h^{Δ^n} conditioned on $h^{\Delta^n}(0) = 0$ and ϕ are positively correlated (see Lemma B.8). Focussing, for simplicity, on (4.67) and abbreviating $\phi'(x) := \phi(x) + \frac{2}{\sqrt{g}}a(x)$, for each $1 \leq k \leq r$ we thus have

$$\nu^0(\phi'(x) \geq 0 : x \in \Delta^r) \geq \nu^0(\phi'(x) \geq 0 : x \in \Delta^k) \nu^0(\phi'(x) \geq 0 : x \in \Delta^r \setminus \Delta^k) \quad (4.69)$$

By inserting the event $\{\tilde{K} \leq k\}$, we then get

$$\begin{aligned} v^0(\phi'(x) \geq 0: x \in \Delta^r \setminus \Delta^k) \\ \geq P\left(\{\tilde{K} \leq k\} \cap \bigcap_{\ell=1}^r \{S_k \geq -\tilde{R}_k(\ell)\} \cap \bigcap_{\ell=k}^r \{S_k \geq \tilde{R}_k(\ell)\}\right), \end{aligned} \quad (4.70)$$

where we also noted that, by (4.21–4.22), on $\{\tilde{K} \leq k\}$ the event on the left is a subset of the first giant intersection on the right. Then we applied (4.17).

Now we use (4.58) conclude that the right-hand side of (4.70) is at least

$$P\left(\bigcap_{\ell=1}^r \{S_k \geq -\tilde{R}_k(\ell)\}\right) P\left(\bigcap_{\ell=k}^r \{S_k \geq \tilde{R}_k(\ell)\} \mid \bigcap_{\ell=1}^r \{S_k \geq -\tilde{R}_k(\ell)\}\right) - \tilde{c}_1 \frac{e^{-\tilde{c}_2(\log k)^2}}{\sqrt{r}}. \quad (4.71)$$

Since $\tilde{R}_k(\ell) \geq 0$, our embedding of the random walk into Brownian motion as detailed in (4.38) shows that the first probability can be bounded as

$$P\left(\bigcap_{\ell=1}^r \{S_k \geq -\tilde{R}_k(\ell)\}\right) \geq P(B_s \geq 0: s \in [1, t_r]) \quad (4.72)$$

where $t_r = \text{Var}(S_r)$. By the Reflection Principle, this is at least a constant times $t_r^{-1/2}$, which by (4.36) is at least a constant times $r^{-1/2}$. The second probability in (4.71) is bounded below using Lemma 4.19 by a quantity of the form $1 - ck^{-\frac{1}{16}}$. The left inequality in (4.67) follows once k is taken sufficiently large.

Concerning the lower bound in (4.66), here we simply set $s := t$ and then proceed by an argument quite analogous to the one above. (The only change is that we need to write $R_k(\ell) + |t|$ instead of $\tilde{R}_k(\ell)$.) We omit further details for brevity. \square

Our final task in this subsection is to show that configurations conditioned to stay above (or below) a function will leave a uniform gap between the conditional and typical value. This gap will be crucial in various approximation arguments in the next section. Here, for ease of expression, we write “ $f \not\geq g$ in Λ ” to designate that there is $x \in \Lambda$ such that $f(x) < g(x)$.

Lemma 4.22 *For all $k \geq 1$ and $\varepsilon > 0$ there is $\delta > 0$ such that for all $r \geq k$ and $s \in \{0, \delta\}$,*

$$v^0\left(\phi + \frac{2}{\sqrt{g}}\mathbf{a} \not\geq \delta - s \text{ in } \Delta^k \setminus \{0\} \mid \phi + \frac{2}{\sqrt{g}}\mathbf{a} \geq -s \text{ in } \Delta^r\right) \leq \varepsilon. \quad (4.73)$$

Similarly, for all $k \geq 1$, all $\varepsilon > 0$ and all $t_0 > 0$ there is $\delta > 0$ such that for all $t \in \mathbb{R}$ with $|t| < t_0$ and all $n \geq k$,

$$P\left(h^{\Delta^n} \not\leq \mathbf{m}_n - \delta \text{ in } \Delta^k \setminus \{0\} \mid h^{\Delta^n} \leq \mathbf{m}_n, h^{\Delta^n}(0) = 0\right) \leq \varepsilon, \quad (4.74)$$

where $\mathbf{m}_n(x) := (m_{2^n} + t)(1 - \mathbf{g}^{\Delta^n})$.

Proof. For convenience, both parts will again use the fact that the laws of the random fields ϕ and h^{Δ^n} are strong-FKG (see Lemma B.8). Our argument will be facilitated by the following general observations: If ϕ is a field on a set Λ with the strong-FKG property and $f: \Lambda \rightarrow \mathbb{R}$ is a function, then

$$P(\phi \not\geq f + \delta \mid \phi \geq f) \leq \sum_{x \in \Lambda} P(\phi(x) < f(x) + \delta \mid \phi(x) \geq f(x)). \quad (4.75)$$

Moreover, if $\phi(x)$ is normal with mean zero and $\sigma^2 := \text{Var}(\phi(x)) > 0$, and $f(x) \leq 0$, then

$$P(\phi(x) \not\geq f(x) + \delta \mid \phi(x) \geq f(x)) \leq \frac{2}{\sqrt{2\pi}} \frac{\delta}{\sigma} \leq \frac{\delta}{\sigma}. \quad (4.76)$$

This follows by a straightforward estimate of the probability density of $\phi(x)$.

The bound (4.73) now follows with the choice $f(x) := -\frac{2}{\sqrt{g}}\mathfrak{a}(x) - s$ and $\varepsilon := 2c\delta|\Delta^k|$, where c is a number such that c^{-2} is the minimum of $\text{Var}(\phi(x))$ over $x \in \Delta^k \setminus \{0\}$. The bound (4.74) is obtained analogously; we just need to also observe that, since $(h^{\Delta^n}(x)|h^{\Delta^n}(0) = 0)$ converges in law to ϕ as $n \rightarrow \infty$, we have $\inf_{n \geq k} \text{Var}(h^{\Delta^n}(x)|h^{\Delta^n}(0) = 0) > 0$ for each $x \in \Delta^k \setminus \{0\}$. \square

4.5 More general outer domains.

In order to simplify exposition, the derivations in the previous sections have been based on the definition of $\{\Delta^k : k \geq 1\}$ via (3.26). However, applications will occasionally require estimates also in the situations when the last domain in the sequence $\Delta^0, \Delta^1, \dots, \Delta^n$ is replaced by a slightly more general domain, albeit of the same spatial scale. Here we observe:

Lemma 4.23 *Let $q \geq 1$. Then there are constants $c_1, c_2 \in (0, \infty)$ such that if we replace Δ^n for $n \geq 1$ by a set $D \subset \mathbb{Z}^2$ satisfying*

$$\Delta^n \subseteq D \subseteq \Delta^{n+q}, \quad (4.77)$$

then

$$c_1 \leq \text{Var}(\varphi_n(0)) \leq c_2. \quad (4.78)$$

Similarly, when we replace Δ^n by any D obeying (4.77), the conclusions of Lemmas 3.7, 3.8 and (3.41) in Lemma 3.9 hold as stated for all $k \leq n$ with the constants that may depend on q but not on n . The same holds for (4.57) in Lemma 4.20, (4.66) in Lemma 4.21 and (4.74) in Lemma 4.22.

Proof. The proofs of the statements under consideration depend sensitively on the underlying domains only via (3.33), which we use only for Δ^k with $k \leq n-1$, and bounds on the variance of $\varphi_k(0)$. The only variance that changes when Δ^n is replaced by above D is that of $\varphi_n(0)$, which is bounded by (3.21) and the fact that $D \mapsto \text{Var}(h^D(0))$ is non-decreasing with respect to set inclusion. The bounds in Lemmas 3.9 and 3.10 use only the various coarse facts about the underlying domain spelled out in Lemmas B.7 and B.12. The statements of Lemmas 4.20, 4.21 and 4.22 then follow as well. \square

5. LIMIT EXTREMAL PROCESS

We are now ready to start proving our main results. We begin by addressing the existence of the limiting distribution of the clusters. The key technical input for these will be the asymptotic formulas stated in Propositions 5.1–5.2 below.

5.1 Key asymptotic formulas.

Abusing the standard notations slightly, let $C_b(\mathbb{R}^{\Delta^j})$ denote the class of bounded and continuous functions $f: \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ that depend only on the coordinates in Δ^j . We will also write $C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2}) :=$

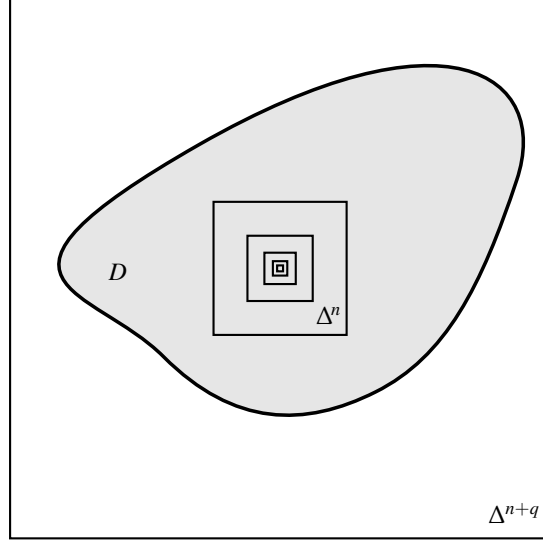


FIG. 7: An illustration of the setting for more general outer domains with parametrization as in (4.77). The set D is lightly shaded, the parameter q equals 2.

$\bigcup_{j \geq 1} C_b(\mathbb{R}^{\Delta_j})$. Given an integer $\ell \geq 1$ and an $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$, define

$$\Xi_\ell^{\text{in}}(f) := E \left(f \left(\phi_\ell + \frac{2}{\sqrt{g}} \mathbf{a} \right) S_\ell \mathbf{1}_{\{S_\ell \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^\ell} \mathbf{1}_{\{\phi_\ell(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) \geq 0\}} \right), \quad (5.1)$$

where

$$\phi_\ell(x) := h^{\Delta^\ell}(x) - h^{\Delta^\ell}(0). \quad (5.2)$$

Given also an integer $n \geq \ell$ and a number $t \in \mathbb{R}$, we also set

$$\Xi_{n,\ell}^{\text{out}}(t) := E \left(S_{n-\ell} \mathbf{1}_{\{S_{n-\ell} \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^n \setminus \Delta^{n-\ell}} \mathbf{1}_{\{h^{\Delta^n}(x) \leq (m_{2^n} + t)(1 - \mathbf{g}^{\Delta^n}(x))\}} \middle| S_{n+1} = 0 \right). \quad (5.3)$$

The key tool for much of our forthcoming derivations are the following two propositions:

Proposition 5.1 *Let $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$. For each $\varepsilon > 0$ there is $\ell_0 \geq 1$ such that for all $\ell \geq \ell_0$ and all $r \geq \ell$ sufficiently large, we have*

$$\left| E_{\nu^0} \left(f \left(\phi + \frac{2}{\sqrt{g}} \mathbf{a} \right) \prod_{x \in \Delta^r} \mathbf{1}_{\{\phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) \geq 0\}} \right) - \frac{1}{\sqrt{\log 2}} \frac{\Xi_\ell^{\text{in}}(f)}{\sqrt{r}} \right| \leq \frac{\varepsilon}{\sqrt{r}}. \quad (5.4)$$

Proposition 5.2 *Let $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$. For each $\varepsilon > 0$ and each $t_0 > 0$ there is $\ell_0 \geq 1$ such that for all $t \in \mathbb{R}$ with $|t| < t_0$, all $\ell \geq \ell_0$ and all n with $\ell \leq n^{1/8}$,*

$$\left| E \left(f(h^{\Delta^n} + \mathbf{m}_n) \mathbf{1}_{\{h^{\Delta^n} \geq -\mathbf{m}_n\}} \middle| h^{\Delta^n}(0) = 0 \right) - \frac{1}{n} \frac{2}{g \log 2} \Xi_\ell^{\text{in}}(f) \Xi_{n,\ell}^{\text{out}}(t) \right| \leq \frac{\varepsilon}{n}, \quad (5.5)$$

where, as before, $\mathbf{m}_n(x) := (m_{2^n} + t)(1 - \mathbf{g}^{\Delta^n}(x))$.

Here, in (5.5), we used the $h^{\Delta^n} \leftrightarrow -h^{-\Delta^n}$ symmetry of the field conditional to $h^{\Delta^n}(0) = 0$ to write the expectation in a form close to that in (5.4). We will now give proofs of these propositions, starting with Proposition 5.1 first. We need two lemmas:

Lemma 5.3 *We have:*

$$\lim_{\ell \rightarrow \infty} \limsup_{r \rightarrow \infty} \sqrt{r} E_{\nu^0} \left(1_{\{S_\ell \leq \ell^{1/6}\}} \prod_{x \in \Delta^r} 1_{\{\phi(x) + \frac{2}{\sqrt{8}} a(x) \geq 0\}} \right) = 0. \quad (5.6)$$

Proof. Thanks to (4.21–4.22) and Lemma 4.20, the expectation is at most

$$\frac{c_1 e^{-c_2(\log \ell)^2}}{\sqrt{r}} + P \left(\{S_\ell \leq \ell^{1/6}\} \cap \bigcap_{j=1}^r \{S_j \geq -\tilde{R}_\ell(j)\} \right). \quad (5.7)$$

Invoking the argument from the proof of Lemma 4.16, the second term is bounded by

$$(1 - e^{-c(\log \ell)^2})^{-1} P^0 \left(\{B_{t_\ell} \leq \ell^{1/6}\} \cap \{B_s \geq -2\zeta(s) : s \in [0, t_r]\} \right), \quad (5.8)$$

where $c \in (0, \infty)$ and $\zeta(s) := C[1 + \log(\ell \vee (s/\sigma_{\min}^2))]$. Lemma A.11 (with $x := \ell^{1/6}$, $u := t_\ell$ and $t := t_r$) now dominates the probability by $c\ell^{-1/6}/\sqrt{r}$. \square

Lemma 5.4 *There is a constant $c \in (0, \infty)$ such that for $\ell \geq 1$ sufficiently large and $r \geq \ell$ sufficiently large,*

$$\left| P \left(\bigcap_{j=\ell+1}^r \{S_j \geq 0\} \mid \sigma(S_\ell) \right) - \frac{1}{\sqrt{\log 2}} \frac{S_\ell}{\sqrt{r}} \right| \leq c \frac{\ell^4}{r} \frac{S_\ell}{\sqrt{r}} \quad (5.9)$$

holds on the event $\{S_\ell \in [\ell^{1/6}, \ell^2]\}$.

Proof. The conditional probability is bounded from below by the probability that the standard Brownian motion started from S_ℓ stays positive for time $t_r - t_\ell$. Since $S_\ell \leq \ell^2$ and t_r is at least a constant times r , Lemma A.1 shows that this probability is at least

$$\sqrt{\frac{2}{\pi}} \frac{S_\ell}{\sqrt{t_r}} (1 - c\ell^4/r) \quad (5.10)$$

for some constant $c \in (0, \infty)$. The lower bound follows from the fact that, thanks to Lemma 3.6, for each $\varepsilon > 0$ we have $t_r \leq (1 + \varepsilon)(g \log 2)r$ as soon as r is sufficiently large.

For the upper bound we use $\{S_j \geq 0\} \subseteq \{S_j \geq -(\log j)^2\}$ for $j = \ell, \dots, r$ and then bound the resulting probability by

$$P(B_{t_j} \geq -\zeta(t_j) : j = \ell, \dots, r \mid B_{t_\ell} = x) \quad \text{evaluated at } x := S_\ell, \quad (5.11)$$

where $\zeta(s) := [\log(s/\sigma_{\max}^2)]^2$ with σ_{\max}^2 denoting the supremum in (4.36). Using the left-hand side of the bound in Lemma 4.16 (integrated with respect to the probability density of the end point over y) this is now bounded by

$$\prod_{j=\ell}^r \left(1 - e^{-2\sigma_{\min}^{-2} [\log(cj)]^2} \right)^{-1} P(B_s \geq -2\zeta(s) : s \in [t_\ell, t_r] \mid B_{t_\ell} = x) \Big|_{x=S_\ell}, \quad (5.12)$$

where $c := \sigma_{\min}^2 / \sigma_{\max}^2$ with σ_{\min}^2 denoting the infimum in (4.36). Invoking Proposition 4.9 with the help of Lemma 4.12, the probability on the right is bounded by

$$(1 + \delta_\ell) \sqrt{\frac{2}{\pi}} \frac{S_\ell}{\sqrt{t_r - t_\ell}}, \quad (5.13)$$

uniformly on $\{S_\ell \in [\ell^{1/6}, \ell^2]\}$, where δ_ℓ is a number such that $\delta_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. The claim again follows by $t_\ell \geq (1 - \varepsilon)(g \log 2)r$. \square

Proof of Proposition 5.1. Let $\varepsilon > 0$ and pick a function $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$. Assume without loss of generality that $0 \leq f \leq 1$ and let k_0 be such that f only depends on the coordinates in Δ^{k_0} and let $\ell \geq k_0$. Our task is to replace ϕ by ϕ_ℓ in the argument of f as well as in the part of the product corresponding to $x \in \Delta^\ell$. For this we first note that

$$\phi_\ell(x) = \sum_{j=0}^{\ell} \left(\mathbf{b}_j(x) \phi_j(0) + \chi_j(x) + h'_j(x) \right). \quad (5.14)$$

For each $j \in \{1, \dots, \ell\}$, we thus get

$$\max_{x \in \Delta^j} |\phi(x) - \phi_\ell(x)| \leq c_1 (\log \ell)^2 2^{(j-\ell)/2} \quad \text{on } \{\tilde{K} \leq \ell\}. \quad (5.15)$$

Moreover, on $\{\tilde{K} \leq k\}$, both ϕ and ϕ_ℓ are bounded on Δ^{k_0} by a k -dependent quantity only. Thanks to uniform continuity of f on compact sets, we thus get

$$\lim_{\ell \rightarrow \infty} \limsup_{r \rightarrow \infty} \sqrt{r} E_{\nu^0} \left(\left| f\left(\phi + \frac{2}{\sqrt{g}} \mathbf{a}\right) - f\left(\phi_\ell + \frac{2}{\sqrt{g}} \mathbf{a}\right) \right| \prod_{x \in \Delta^r} 1_{\{\phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x) \geq 0\}} \right) = 0 \quad (5.16)$$

by inserting the indicator of $\{\tilde{K} \leq k\}$ for $k \geq k_0$, applying (5.15) for $\ell \geq k$, invoking (4.21–4.22) in conjunction with (4.58) and taking the stated limits followed by $k \rightarrow \infty$.

Having brought the argument of f to what it is supposed to be, for each $k \geq 1$ let $\delta_k > 0$ be such that (4.73) holds with $\delta := \delta_k$ and the above ε . Abbreviate $\phi'(x) := \phi(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x)$ and define

$$A_{r,\ell,k} := \{\tilde{K} \leq k\} \cap \{\phi' \geq \delta_k \text{ on } \Delta^k\} \cap \{S_\ell \in [\ell^{1/6}, \ell^2]\} \cap \bigcap_{j=k+1}^r \{S_j \geq 2\tilde{R}_k(j)\}. \quad (5.17)$$

Lemmas 4.20, 4.22, 4.18 and 5.3 show

$$\limsup_{r \rightarrow \infty} \sqrt{r} E_{\nu^0} \left(1_{A_{r,\ell,k}^c} \prod_{x \in \Delta^r} 1_{\{\phi'(x) \geq 0\}} \right) \leq \varepsilon + ck^{-\frac{1}{16}} < 2\varepsilon \quad (5.18)$$

once k and $\ell \geq k$ are sufficiently large. We fix such a k for the rest of the argument.

Using the shorthand $\phi'_\ell(x) := \phi_\ell(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x)$, the bound in (5.18) yields

$$\begin{aligned} & E_{\nu^0} \left(f(\phi'_\ell) \prod_{x \in \Delta^r} 1_{\{\phi'(x) \geq 0\}} \right) \\ & \leq \frac{2\varepsilon}{\sqrt{r}} + E_{\nu^0} \left(f(\phi'_\ell) 1_{\{\tilde{K} \leq k\}} 1_{\{S_\ell \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^k} 1_{\{\phi'(x) \geq \delta_k\}} \prod_{j=k+1}^r 1_{\{S_j \geq 2\tilde{R}_k(j)\}} \right). \end{aligned} \quad (5.19)$$

Now pick $\ell \geq k$ so large that the right-hand side of (5.15) is less than δ_k for $j = 1, \dots, k$ and less than $\tilde{R}_k(j)$ for $j = k+1, \dots, \ell$. (This assumes that the constant C defining $\tilde{R}_k(j)$ was taken large enough.) Then

$$\{\phi' \geq \delta_k \text{ on } \Delta^k\} \subseteq \{\phi'_\ell \geq 0 \text{ on } \Delta^k\} \quad (5.20)$$

while, for $j = k + 1, \dots, \ell$,

$$\{\tilde{K} \leq k\} \cap \{S_j \geq 2\tilde{R}_k(j)\} \subseteq \{\phi' \geq \tilde{R}_k(j) \text{ on } \Delta^j \setminus \Delta^{j-1}\} \subseteq \{\phi'_\ell \geq 0 \text{ on } \Delta^j \setminus \Delta^{j-1}\}. \quad (5.21)$$

It follows that the expectation on the right of (5.19) is bounded by

$$E_{\nu^0} \left(f(\phi'_\ell) 1_{\{S_\ell \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^\ell} 1_{\{\phi'_\ell(x) \geq 0\}} \prod_{j=\ell+1}^r 1_{\{S_j \geq 0\}} \right). \quad (5.22)$$

Conditional on S_ℓ , the field ϕ'_ℓ is independent of $\sigma(S_{\ell+1}, \dots, S_r)$. Lemma 5.4 and the Bounded Convergence Theorem then yield

$$\lim_{\ell \rightarrow \infty} \limsup_{r \rightarrow \infty} \left| \sqrt{r} E_{\nu^0} \left(f(\phi'_\ell) 1_{\{S_\ell \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^\ell} 1_{\{\phi'_\ell(x) \geq 0\}} \prod_{j=\ell+1}^r 1_{\{S_j \geq 0\}} \right) - \Xi_\ell^{\text{in}}(f) \right| = 0. \quad (5.23)$$

In conjunction with (5.19) and the derivations above, this proves one “half” of (5.4).

To get the other “half” of (5.4), we in turn define

$$\tilde{A}_{r,\ell,k} := \{\tilde{K} \leq k\} \cap \{\phi' \geq 0 \text{ on } \Delta^r\} \cap \{S_\ell \in [\ell^{1/6}, \ell^2]\} \quad (5.24)$$

and use Lemmas 4.20, 4.22, 4.18 and 5.3 to get, for $k \leq \ell$ large enough,

$$\limsup_{r \rightarrow \infty} \sqrt{r} E \left(1_{\tilde{A}_{r,\ell,k}^c} \prod_{x \in \Delta^k} 1_{\{\phi'(x) \geq -\delta_k\}} \prod_{j=k+1}^r 1_{\{S_j \geq -2\tilde{R}_k(j)\}} \right) < 2\varepsilon. \quad (5.25)$$

Now, clearly,

$$\begin{aligned} E_{\nu^0} \left(f(\phi'_\ell) \prod_{x \in \Delta^r} 1_{\{\phi'(x) \geq 0\}} \right) &\geq E_{\nu^0} \left(f(\phi'_\ell) 1_{\tilde{A}_{r,\ell,k}} \right) \\ &\geq E_{\nu^0} \left(f(\phi'_\ell) 1_{\tilde{A}_{r,\ell,k}} \prod_{x \in \Delta^k} 1_{\{\phi'(x) \geq -\delta_k\}} \prod_{j=k+1}^r 1_{\{S_j \geq -2\tilde{R}_k(j)\}} \right). \end{aligned} \quad (5.26)$$

Assuming $\ell \gg k$, on $\{\tilde{K} \leq k\}$, which is a subset of $\tilde{A}_{r,\ell,k}$, we have $\{\phi' \geq -\delta_k \text{ on } \Delta^k\} \supseteq \{\phi'_\ell \geq 0 \text{ on } \Delta^k\}$ and $\{S_j \geq -2\tilde{R}_k(j)\} \supseteq \{\phi'_\ell \geq 0 \text{ on } \Delta^j \setminus \Delta^{j-1}\}$ for $j = k + 1, \dots, \ell$. By (5.25),

$$\begin{aligned} E_{\nu^0} \left(f(\phi'_\ell) \prod_{x \in \Delta^r} 1_{\{\phi'(x) \geq 0\}} \right) &\geq E_{\nu^0} \left(f(\phi'_\ell) 1_{\tilde{A}_{r,\ell}} \prod_{x \in \Delta^\ell} 1_{\{\phi'_\ell(x) \geq 0\}} \prod_{j=\ell+1}^r 1_{\{S_j \geq 0\}} \right) \\ &\geq -\frac{2\varepsilon}{\sqrt{r}} + E_{\nu^0} \left(f(\phi'_\ell) 1_{\{S_\ell \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in \Delta^\ell} 1_{\{\phi'_\ell(x) \geq 0\}} \prod_{j=\ell+1}^r 1_{\{S_j \geq 0\}} \right). \end{aligned} \quad (5.27)$$

The proof of (5.4) is now finished using (5.23). \square

In order to prove Proposition 5.2, we need substitutes for Lemmas 5.3–5.4:

Lemma 5.5 *Recall the shorthand $\mathfrak{m}_n(x) := (m_{2^n} + t)(1 - \mathfrak{g}^{\Delta^n}(x))$. Then*

$$\lim_{\ell \rightarrow \infty} \limsup_{n \rightarrow \infty} n E \left(1_{\{S_\ell \wedge S_{n-\ell} \leq \ell^{1/6}\}} \prod_{x \in \Delta^n} 1_{\{h^{\Delta^n}(x) \geq -\mathfrak{m}_n(x)\}} \middle| h^{\Delta^n}(0) = 0 \right) = 0. \quad (5.28)$$

Lemma 5.6 *There is a constant $c \in (0, \infty)$ such that*

$$\left| P \left(\bigcap_{j=\ell+1}^{n-\ell} \{S_j \geq 0\} \middle| \sigma(S_\ell, S_{n-\ell}) \right) - \frac{2}{g \log 2} \frac{S_\ell S_{n-\ell}}{n} \right| \leq c \frac{\ell^4}{n} \frac{S_\ell S_{n-\ell}}{n} \quad (5.29)$$

holds on $\{S_\ell, S_{n-\ell} \in [\ell^{1/6}, \ell^2]\}$.

Proofs of Lemmas 5.5–5.6. These statements are proved by arguments nearly identical to those of Lemmas 5.3–5.4. The following two points are worthy of a note: First, a bound in Lemma A.11 can be used for Brownian motion conditioned on $B_t = 0$ (and a slightly worse numerical constant). This follows from the decoupling argument in Lemma A.5. Second, the constant multiplying $S_\ell S_{n-\ell}$ is different from that multiplying S_ℓ in Lemma 5.3. This stems from the difference of the prefactors in (A.4) and (A.3). Further details are left to the reader. \square

Proof of Proposition 5.2. Pick $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$ and assume without loss of generality that $1 \leq f \leq 2$. By routine approximation arguments we can suppose that f is in fact Lipschitz in the variables that it depends on. Similarly as in the proof above, on $\{K \leq k\} \cap \{h^{\Delta^n}(0) = 0\}$ we have

$$\max_{x \in \Delta^j} |h^{\Delta^n}(x) - \phi_\ell(x)| \leq c_1 (\log \ell)^2 2^{-(\ell-k)/2}, \quad j = k, \dots, \ell \quad (5.30)$$

for each $k \leq \ell \leq n/2$. This and the fact that $\mathbf{m}_n(x) \rightarrow \frac{2}{\sqrt{g}} \mathbf{a}(x)$ as $n \rightarrow \infty$ (uniformly on compact intervals of t) imply that it suffices to prove (5.5) for $f(h^{\Delta^n} + \mathbf{m}_n)$ replaced by $f(\phi'_\ell)$, where

$$\phi'_\ell(x) := \phi_\ell(x) + \frac{2}{\sqrt{g}} \mathbf{a}(x), \quad (5.31)$$

for any ℓ sufficiently large. Here we again used the inclusions in (4.21–4.22) in conjunction with (4.58) to bound the contribution of the event when the control variable K is large.

Let P^0 , resp., E^0 abbreviate the probability, resp., expectation conditional on $\{h^{\Delta^n}(0) = 0\}$. Given $\varepsilon > 0$ and $k \geq 1$, let δ_k be such that (4.74) holds with $\delta := 2\delta_k$ for this ε . For ℓ with $k \leq \ell \leq n/2$, define

$$\begin{aligned} A_{n,\ell,k} &:= \{K \leq k\} \cap \{h^{\Delta^n} \geq -\mathbf{m}_n + 2\delta_k \text{ on } \Delta^k\} \\ &\quad \cap \{S_\ell, S_{n-\ell} \in [2\ell^{1/6}, \frac{1}{2}\ell^2]\} \cap \bigcap_{j=k+1}^{n-\ell} \{S_j \geq 2R_k(j)\}. \end{aligned} \quad (5.32)$$

By Lemmas 4.20 and 4.22, Lemma 4.19 and Lemma 5.5 we have

$$\limsup_{n \rightarrow \infty} n E^0 \left(1_{A_{n,\ell,k}^c} \prod_{x \in \Delta^n} 1_{\{h^{\Delta^n}(x) \geq -\mathbf{m}_n(x)\}} \right) < 2\varepsilon \quad (5.33)$$

once k is large enough. We again fix this k throughout the rest of the argument.

For n sufficiently large, we now get

$$\begin{aligned} &E^0 \left(f(\phi'_\ell) \prod_{x \in \Delta^n} 1_{\{h^{\Delta^n}(x) \geq -\mathbf{m}_n(x)\}} \right) \\ &\leq \frac{2\varepsilon}{n} + E^0 \left(f(\phi'_\ell) 1_{\{S_\ell, S_{n-\ell} \in [2\ell^{1/6}, \frac{1}{2}\ell^2]\}} \prod_{x \in \Delta^k} 1_{\{\phi'_\ell \geq 2\delta_k\}} \right. \\ &\quad \left. \times \prod_{j=k+1}^{n-\ell} 1_{\{S_j \geq 2R_k(j)\}} \prod_{x \in \Delta^n \setminus \Delta^{n-\ell}} 1_{\{h^{\Delta^n}(x) \geq -\mathbf{m}_n(x)\}} \right). \end{aligned} \quad (5.34)$$

The next important observation is that, conditional on

$$\mathcal{F}_\ell := \sigma(S_1, \dots, S_\ell, S_{n-\ell}, \dots, S_{n+1}) \quad (5.35)$$

the field ϕ_ℓ , the random variables $\{S_j: j = \ell, \dots, n-\ell\}$ and the field $\{h^{\Delta^n}(x): x \in \Delta^n \setminus \Delta^{n-\ell}\}$ are independent under P^0 . Using Lemma 5.6, the expectation on the right of (5.34) is bounded by

$$\begin{aligned} & \frac{1}{n} \left(\frac{2}{g \log 2} + c \frac{\ell^4}{n} \right) E^0 \left(f(\phi'_\ell) 1_{\{S_\ell, S_{n-\ell} \in [\ell^{1/6}, \frac{1}{2}\ell^2]\}} S_\ell S_{n-\ell} \prod_{x \in \Delta^k} 1_{\{\phi'_\ell(x) \geq 2\delta_k\}} \right. \\ & \quad \times \left. \prod_{j=k+1}^\ell 1_{\{S_j \geq 2R_k(j)\}} \prod_{x \in \Delta^n \setminus \Delta^{n-\ell}} 1_{\{h^{\Delta^n}(x) \geq -m_n(x)\}} \right). \end{aligned} \quad (5.36)$$

Our task is to dominate the expectation by $(1 + o(1)) \Xi_\ell^{\text{in}}(f) \Xi_{n,\ell}^{\text{out}}(t)$. The main issue is to decouple the scales $j \leq \ell$ from the scales $j \geq n - \ell$.

First we recall that, by (3.53–3.54), for each $r = 0, \dots, n$ the random variables

$$\tilde{S}_j^{(r)} := S_j - \left(\sum_{i=0}^{j-1} c_{r-1}(i) \right) S_r, \quad j = 1, \dots, r, \quad (5.37)$$

where $c_n(k)$ are as in (3.52), as well as the field

$$\tilde{\phi}_r(x) := \phi_r(x) - \left(\sum_{j=0}^{r-1} b_j(x) c_{r-1}(j) \right) S_r, \quad (5.38)$$

are independent of $\sigma(S_r, S_{r+1}, \dots, S_{n+1})$ under P^0 . Lemma 3.6 shows that $0 \leq c_n(k) \leq \tilde{c}/n$ for some $\tilde{c} \in (0, \infty)$ and Lemma 3.7 then ensures that, for some $c \in (0, \infty)$ and all $k = 0, \dots, n$,

$$\left| \sum_{j=0}^n b_j(x) c_n(j) \right| \leq c \frac{r}{n}, \quad x \in \Delta^r. \quad (5.39)$$

It follows that, on $\{S_{n-\ell} \in [\ell^{1/6}, \ell^2]\}$ and for $\ell < n^{1/4}$,

$$\max_{x \in \Delta^\ell} |\phi'_\ell(x) - \tilde{\phi}'_{n-\ell}(x)| \leq c \ell^3 (n - \ell)^{-1} \leq 2c \ell^3 n^{-1} \quad (5.40)$$

and

$$0 \leq S_j - \tilde{S}_j^{(n-\ell)} \leq \tilde{c} \ell^3 n^{-1}, \quad j = 0, \dots, \ell. \quad (5.41)$$

Thanks to the Lipschitz property of f , this permits us to bound the expectation in (5.36) by a quantity of the form $(1 + O(n^{-1/4}))$ — where the ℓ^4 term comes from bounding $S_\ell S_{n-\ell}$ by their worst case value — times

$$\begin{aligned} & E^0 \left(f(\tilde{\phi}'_{n-\ell}) 1_{\{\tilde{S}_\ell^{(n-\ell)} \in [\ell^{1/6}, \frac{1}{2}\ell^2]\}} \tilde{S}_\ell^{(n-\ell)} \prod_{x \in \Delta^k} 1_{\{\tilde{\phi}'_{n-\ell}(x) \geq \delta_k\}} \prod_{j=k+1}^\ell 1_{\{\tilde{S}_j^{(n-\ell)} \geq R_k(j)\}} \right. \\ & \quad \times \left. 1_{\{S_{n-\ell} \in [\ell^{1/6}, \ell^2]\}} S_{n-\ell} \prod_{x \in \Delta^n \setminus \Delta^{n-\ell}} 1_{\{h^{\Delta^n}(x) \geq m_n(x)\}} \right). \end{aligned} \quad (5.42)$$

Conditional on $S_{n-\ell}$, the “small scale” part of the function under expectation is now independent of the rest and so this equals $\Xi_{n,\ell}^{\text{out}}(t)$ times

$$E^0 \left(f(\tilde{\phi}'_{n-\ell}) 1_{\{\tilde{S}_\ell^{(n-\ell)} \in [\ell^{1/6}, \frac{1}{2}\ell^2]\}} \tilde{S}_\ell^{(n-\ell)} \prod_{x \in \Delta^k} 1_{\{\tilde{\phi}'_{n-\ell}(x) \geq \delta_k\}} \prod_{j=k+1}^\ell 1_{\{\tilde{S}_j^{(n-\ell)} \geq R_k(j)\}} 1_{\{S_{n-\ell} \in [\ell^{1/6}, \ell^2]\}} \right). \quad (5.43)$$

The last indicator now permits us to return all $\tilde{\phi}'_{n-\ell}$ back to ϕ'_ℓ and all $\tilde{S}_j^{(n-\ell)}$ back to S_j at the cost of another multiplicative factor $(1 + O(\ell^5 n^{-1}))$. Dropping the indicator and applying the argument (5.19–5.22) then yields the expectation in the definition of $\Xi_\ell^{\text{in}}(f)$.

The complementary (lower) bound is proved by modifications similar to those used in the proof of Proposition 5.1. We omit the details. \square

5.2 Extracting the cluster law.

We are now ready to harvest the first fruits of our hard work in the previous sections. In particular, we will establish the existence of the cluster law and prove the asymptotic in Theorem 2.4. We begin by noting that the estimates in Lemma 4.21 imply uniform bounds on the quantities $\Xi_\ell^{\text{in}}(1)$ and $\Xi_{n,\ell}^{\text{out}}(t)$ from (5.1–5.3).

Lemma 5.7 *There are $c_1, c_2 \in (0, \infty)$ such that for all sufficiently large ℓ ,*

$$c_1 < \Xi_\ell^{\text{in}}(1) < c_2. \quad (5.44)$$

Moreover, for each $t_0 > 0$ there are $c'_1, c'_2 \in (0, \infty)$ such that for all sufficiently large ℓ , all $n \geq \ell^8$ and all $t \in \mathbb{R}$ with $|t| < t_0$, we also have

$$c'_1 < \Xi_{n,\ell}^{\text{out}}(t) < c'_2. \quad (5.45)$$

Proof. Combining (5.4) for $f := 1$ with (4.67) directly yields (5.44). A similar reasoning then shows that (5.5) and (4.66) imply (5.45). \square

With these bounds in hand, Proposition 5.1 readily yields the asymptotic of the probability of the conditional event that leads to the definition of the cluster law:

Proof of Theorem 2.4. The bound in (5.4) reads

$$\sqrt{r} v^0 \left(\phi_x + \frac{2}{\sqrt{g}} \mathbf{a}(x) \geq 0 : |x| \leq 2^r \right) = \frac{\Xi_\ell^{\text{in}}(1)}{\sqrt{\log 2}} + \varepsilon_r(\ell), \quad (5.46)$$

where $\lim_{\ell \rightarrow \infty} \limsup_{r \rightarrow \infty} |\varepsilon_r(\ell)| = 0$. Since the left-hand side is independent of ℓ , taking $r \rightarrow \infty$ followed by $\ell \rightarrow \infty$ along suitable subsequences shows that the *limes superior* of the left-hand side is less than the *limes inferior* of the right-hand side and, similarly, the *limes inferior* of the left-hand side is at least the *limes superior* of the right-hand side. It follows that both limits in

$$\lim_{r \rightarrow \infty} \sqrt{r} v^0 \left(\phi_x + \frac{2}{\sqrt{g}} \mathbf{a}(x) \geq 0 : |x| \leq 2^r \right) = \lim_{\ell \rightarrow \infty} \frac{\Xi_\ell^{\text{in}}(1)}{\sqrt{\log 2}} \quad (5.47)$$

exist and are related as stated. By (5.44) the right-hand side is also positive and finite. The claim then follows with $\tilde{c}^* := \lim_{\ell \rightarrow \infty} \Xi_\ell^{\text{in}}(1)$ (which we denote by $\Xi_\infty^{\text{in}}(1)$ later) by noting that, thanks to monotonicity in r of the probability in (2.10) and the slowly-varying nature of $r \mapsto \log r$, it suffices to prove the desired asymptotic only for radii varying along powers of 2. \square

Proposition 5.1 permits us to work with a large class of test functions f . This in particular permits us to prove the existence of the limit in (2.9). (We state this as a separate proposition because we will only identify the limit measure with that in Theorem 2.1 later.)

Proposition 5.8 (Existence of cluster law) *For every $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$, the limit*

$$\Xi_\infty^{\text{in}}(f) := \lim_{\ell \rightarrow \infty} \Xi_\ell^{\text{in}}(f) \quad (5.48)$$

exists and is finite. Moreover, also the following limit exists and obeys

$$\lim_{r \rightarrow \infty} \frac{E_{\nu^0}(f(\phi + \frac{2}{\sqrt{g}}\mathbf{a}) \prod_{x: |x| \leq r} 1_{\{\phi(x) + \frac{2}{\sqrt{g}}\mathbf{a}(x) \geq 0\}})}{\nu^0(\phi(x) + \frac{2}{\sqrt{g}}\mathbf{a}(x) \geq 0: |x| \leq r)} = \frac{\Xi_{\infty}^{\text{in}}(f)}{\Xi_{\infty}^{\text{in}}(1)}. \quad (5.49)$$

In addition, there is a probability measure ν on $[0, \infty)^{\mathbb{Z}^d}$ such that the limit equals $E_{\nu}(f(\phi))$. This measure has finite level sets almost surely; in fact, for any $c \in (0, \infty)$,

$$\nu(\{\phi \not\geq c(\log k)^2 \text{ on } \Delta^k \setminus \Delta^{k-1}\} \text{ i.o.}) = 0. \quad (5.50)$$

Proof. Abbreviate $\phi'(x) := \phi(x) + \frac{2}{\sqrt{g}}\mathbf{a}(x)$. By Proposition 5.1 we have

$$\frac{E_{\nu^0}(f(\phi') \prod_{x \in \Delta^r} 1_{\{\phi'(x) \geq 0\}})}{\nu^0(\phi'(x) \geq 0: x \in \Delta^r)} = \frac{\Xi_{\ell}^{\text{in}}(f) + \tilde{\varepsilon}_r(\ell)}{\Xi_{\ell}^{\text{in}}(1) + \varepsilon_r(\ell)} \quad (5.51)$$

where $\lim_{\ell \rightarrow \infty} \limsup_{r \rightarrow \infty} |\tilde{\varepsilon}_r(\ell)| = 0$ and similarly for $\varepsilon_r(\ell)$. Since the left hand side does not depend on ℓ while the only dependence on r of the right hand side comes through $\tilde{\varepsilon}_r(\ell)$ and $\varepsilon_r(\ell)$ which vanish as $r \rightarrow \infty$, the argument from the previous proof shows that both sides converge. That argument also gave the existence of the positive limit $\lim_{\ell \rightarrow \infty} \Xi_{\ell}^{\text{in}}(1)$ and so also the limit in (5.48) exists for all $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$. This proves (5.49) for r running along powers of 2; for the general $r \rightarrow \infty$ we then assume $f \geq 0$ and argue by monotonicity and existence of the limit in (2.10) (already proved above).

The linear functional $f \mapsto \xi(f) := \Xi_{\infty}^{\text{in}}(f)/\Xi_{\infty}^{\text{in}}(1)$ is positive and so, when restricted $C_b(\mathbb{R}^{\Delta^j})$, continuous and bounded in the supremum norm. By the Riesz-Markov-Kakutani representation theorem, there is a regular Borel measure ν_j on \mathbb{R}^{Δ^j} such that $\xi(f) = \int \nu_j(d\phi) f(\phi)$ for each $f \in C_b^{\text{loc}}(\mathbb{R}^{\Delta^j})$. In order to show that ν_j is a probability measure, we have to prove tightness. Here we note that (4.17) and the definition of $R_{\tilde{K}}(j)$ imply

$$\{\phi' \not\geq 2k^2 \text{ on } \Delta^j\} \subseteq \{\tilde{K} > k\}, \quad 1 \leq j \leq k, \quad (5.52)$$

once k is sufficiently large. Lemmas 4.20 and 4.18 then show

$$\nu^0(\phi' \not\geq 2k^2 \text{ on } \Delta^j \mid \phi'(x) \geq 0: x \in \Delta^r) \leq ce^{-c'(\log k)^2}. \quad (5.53)$$

Standard approximation arguments extend this bound to the limit measure; taking $k \rightarrow \infty$ then shows $\nu_j(\mathbb{R}^{\Delta^j}) = 1$. The measures $\{\nu_j: j \geq 1\}$ are clearly consistent; the Kolmogorov Extension Theorem then ensures that they are restrictions of a unique probability measure ν on $\mathbb{R}^{\mathbb{Z}^2}$. This measure obeys $\xi(f) = E_{\nu}(f(\phi))$ for all $f \in C_b^{\text{loc}}(\mathbb{R}^{\mathbb{Z}^2})$ as desired.

It remains to prove (5.50). For this we note

$$\{\phi' \not\geq c(\log k)^2 \text{ on } \Delta^k \setminus \Delta^{k-1}\} \cap \{K \leq k\} \subseteq \{S_k \leq \tilde{c}[1 + (\log k)^2]\}. \quad (5.54)$$

Lemmas 4.20 and 4.18 then show that, for any $k \leq k' \leq r$ with k sufficiently large,

$$\nu^0\left(\bigcup_{j=k}^{k'} \{\phi' \not\geq c(\log k)^2 \text{ on } \Delta^k \setminus \Delta^{k-1}\} \mid \phi'(x) \geq 0: x \in \Delta^r\right) \leq c'k^{-\frac{1}{16}} \quad (5.55)$$

once r is large. Taking $r \rightarrow \infty$ followed by $k \rightarrow \infty$ then yield the claim. \square

5.3 Limit of full extreme process.

We will now move to the proof of the distributional convergence in Theorem 2.1 including the characterization of the cluster law in Theorem 2.3. This will be done modulo the proof of a technical Proposition 5.10 which is deferred to the next subsection.

Let $D \in \mathcal{D}$. Given a Radon measure η on $\bar{D} \times \mathbb{R} \times \mathbb{R}^d$ and a measurable function $f: \bar{D} \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$, we will abbreviate

$$\langle \eta, f \rangle := \int \eta(dx dh d\phi) f(x, h, \phi). \quad (5.56)$$

Let D_N be related to D as in (2.1–2.2) and let h^{D_N} be the DGFF in D_N . Recall the notation

$$\Gamma_N^D(t) := \{x: h^{D_N}(x) \geq m_N - t\} \quad (5.57)$$

and, using the notation $\Lambda_r(x)$ from (1.4), write

$$\Theta_{N,r}^D := \{x \in D_N: h^{D_N}(x) = \max_{z \in \Lambda_r(x)} h^{D_N}(z)\} \quad (5.58)$$

for the set of points in D_N that are r -local extrema. Our starting point is the following lemma:

Lemma 5.9 *For any $r_N \rightarrow \infty$ with $N/r_N \rightarrow \infty$ and any continuous $f: \bar{D} \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$ with compact support,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} \max_{M: r \leq M \leq N/r} \left| E(e^{-\langle \eta_{N,r_N}^D, f \rangle}) - E(e^{-\langle \eta_{N,M}^D, f \rangle}) \right| = 0. \quad (5.59)$$

Proof. By assumption $f(x, h, \phi) = 0$ is zero unless $|h| \leq \lambda$, for some $\lambda > 0$. It follows that $\langle \eta_{N,r_N}^D, f \rangle \neq \langle \eta_{N,M}^D, f \rangle$ for some M between r and N/r implies $(\Theta_{N,r_N}^D \triangle \Theta_{N,M}^D) \cap \Gamma_N(\lambda) \neq \emptyset$. But this in turn forces the existence of two local maxima in $\Gamma_N(\lambda)$ that are farther than $M \wedge r_N$ and yet closer than $M \vee r_N$. Hence, for N so large that $r_N \geq r$ and $r_N \leq N/r$,

$$\begin{aligned} \max_{M: r \leq M \leq N/r} \left| E(e^{-\langle \eta_{N,r_N}^D, f \rangle}) - E(e^{-\langle \eta_{N,M}^D, f \rangle}) \right| \\ \leq P(\exists x, y \in \Gamma_N^D(\lambda): r \leq |x - y| \leq N/r). \end{aligned} \quad (5.60)$$

The right-hand side tends to zero in the stated limits by Lemma B.11. \square

Thanks to Lemma 5.9, we may as well pick any $M \in \{r, \dots, \lfloor N/r \rfloor\}$ and work with $\eta_{N,M}^D$ in place of η_{N,r_N}^D . We will choose

$$M = M(N, r) := \max\{2^n: 2^n \leq N/r\}. \quad (5.61)$$

Our next step is to introduce an auxiliary process

$$\hat{\eta}_{N,M}^D := \sum_{x \in D_N} 1_{\{h_x^{D_N} = \max_{z \in \Lambda_M(x)} (h_x^{D_N} - \Phi_z^{M,x})\}} \delta_{x/N} \otimes \delta_{h_x^{D_N} - m_N} \otimes \delta_{\{h_x^{D_N} - h_{x+z}^{D_N} + \Phi_{x+z}^{M,x}: z \in \mathbb{Z}^2\}}, \quad (5.62)$$

where, using the notation $H^D(x, y)$ for the harmonic measure from x to y in D ,

$$\Phi^{M,x}(z) := \sum_{y \in D_N \cap \partial \Lambda_M(x)} H^{(D_N \cap \Lambda_M(x)) \setminus \{x\}}(z, y) h^{D_N}(y). \quad (5.63)$$

Note that $\Phi^{M,x} = h^{D_N}(x)$ outside $\Lambda_M(x)$ and also $\Phi^{M,x}(x) = 0$ because $y := x$ is not included in the sum. As it turns out, the laws of the processes $\eta_{N,M}^D$ and $\hat{\eta}_{N,M}^D$ are very close:

Proposition 5.10 *For any $f: \overline{D} \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, \infty)$ that is continuous with compact support and depends only on a finite number of coordinates of ϕ ,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} \left| E \left(e^{-\langle \eta_{N,M(N,r)}^D, f \rangle} \right) - E \left(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, f \rangle} \right) \right| = 0. \quad (5.64)$$

We defer the proof of this proposition to the next subsection and instead proceed with the proof of the point process convergence. There are several reasons why $h^{D_N} - \Phi^{M,\cdot}$ is more convenient to work with than h^{D_N} . All of them can be deduced from the following observation:

Lemma 5.11 *Suppose $x \in D_N$ is such that $\Lambda_M(x) \subset D_N$ and let $n \in \mathbb{N}$ be such that $M = 2^n$. Consider the σ -algebra*

$$\mathcal{F}_{M,x} := \sigma(h^{D_N}(z) : z \in \{x\} \cup \Lambda_M(x)^c). \quad (5.65)$$

Then for Lebesgue almost every $t \in \mathbb{R}$,

$$\begin{aligned} P(h^{D_N}(x + \cdot) - \Phi^{M,x}(x + \cdot) \in \cdot \mid \mathcal{F}_{M,x}) \\ = P(h^{\Delta^n \setminus \{0\}} + t \mathbf{g}^{\Delta^n} \in \cdot) = P(h^{\Delta^n} \in \cdot \mid h^{\Delta^n}(0) = t) \quad \text{on } \{h^{D_N}(x) = t\}. \end{aligned} \quad (5.66)$$

In particular, conditional on $\sigma(h^{D_N}(x))$, the conditional probability on the left-hand side is independent of $\sigma(h^{D_N}(z) : z \notin \Lambda_M(x))$.

Proof. Our choice of M ensures $\Lambda_M(0) = \Delta^n$. By the Gibbs-Markov property (Lemma B.6) and the fact that $H^{(D_N \cap \Lambda_M(x)) \setminus \{x\}}(x + z, x) = \mathbf{g}^{\Delta^n}(z)$ we have

$$\Phi^{M,x}(x + z) + \mathbf{g}^{\Delta^n}(z) h^{D_N}(x) = E(h^{D_N}(x + z) \mid \sigma(h^{D_N}(y) : y \in \{x\} \cup \Lambda_M(x)^c)). \quad (5.67)$$

Under the assumption that $\Lambda_M(x) \subset D_N$ we thus get

$$h^{D_N}(x + \cdot) - \Phi^{M,x}(x + \cdot) - \mathbf{g}^{\Delta^n}(\cdot) h^{D_N}(x) \stackrel{\text{law}}{=} h^{\Delta^n \setminus \{0\}}(\cdot). \quad (5.68)$$

Conditional on $\mathcal{F}_{M,x}$ the last two terms on the left-hand side are effectively constant, with the very last one equal to $\mathbf{g}^{\Delta^n}(\cdot) t$ on $\{h^{D_N}(0) = t\}$. The claim follows by noting, as observed in the proof of Lemma 3.2, $h^{\Delta^n \setminus \{0\}} + t \mathbf{g}^{\Delta^n}$ has the law of h^{Δ^n} conditioned on $h^{\Delta^n}(0) = t$. \square

The role of the auxiliary process is to arrange that, after conditioning on the position and value at each relevant local maximum, the “clusters” associated with distinct local maxima are independent. As foreseen in (3.1–3.2), the following law naturally arises,

$$\mathbf{v}^{(M,t)}(\cdot) := P(h^{\Delta^n}(0) - h^{\Delta^n} \in \cdot \mid h^{\Delta^n}(0) = m_N + t, h^{\Delta^n} \leq h^{\Delta^n}(0)), \quad (5.69)$$

where $n \in \mathbb{N}$ is such that $M = 2^n$. Indeed, we have:

Lemma 5.12 *Let $f = f(x, h, \phi) : D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \rightarrow [0, \infty)$ be continuous with compact support and depending only on $\{\phi(x) : x \in \Lambda_M(x)\}$. Define $\tilde{f}_{N,r} : D \times \mathbb{R} \rightarrow [0, \infty)$ by*

$$e^{-\tilde{f}_{N,r}(x,h)} := E_{\mathbf{v}^{(M,h)}}(e^{-f(x,h,\phi)}), \quad (5.70)$$

where $M = M(N, r)$ is as above. Then there is $r_0 = r_0(f)$ such that for all $r \geq r_0$ and all N sufficiently large,

$$E(e^{-\langle \hat{\eta}_{N,M}^D, f \rangle}) = E(e^{-\langle \hat{\eta}_{N,M}^D, \tilde{f}_{N,r} \rangle}). \quad (5.71)$$

Proof. Recall our notation $\Theta_{N,r}^D$ for the set of sites in D_N where h^{D_N} has an r -local maximum and let $\widehat{\Theta}_{N,M}^D$ analogously denote the set

$$\widehat{\Theta}_{N,M}^D := \{x \in D_N : \max_{z \in \Lambda_M(x)} h^{D_N}(z) - \Phi^{M,x}(z) = h^{D_N}(x)\}. \quad (5.72)$$

Using inclusion-exclusion, we then get

$$\begin{aligned} E(e^{-\langle \widehat{\eta}_{N,M}^D, f \rangle}) &= 1 \\ &+ \sum_{n=1}^{|D_N|} \sum_{\substack{A \subset D_N \\ |A|=n}} E \left(\prod_{x \in A} \left((e^{-f(x/N, h^{D_N}(x) - m_N, h^{D_N}(x) - h^{D_N}(x+\cdot) + \Phi^{M,x}(x+\cdot))} - 1) 1_{\{x \in \widehat{\Theta}_{N,r}^D\}} \right) \right), \end{aligned} \quad (5.73)$$

where the sum actually terminates at the maximal number of distinct translates of $\Lambda_M(0)$ one can center at points of D_N so that each center point belongs to only one of these sets.

Since h^{D_N} is continuously distributed, the collection of sets $\{\Lambda_M(x) : x \in A\}$ is a.s. disjoint for any A contributing (non-trivially) to the above sum. Thanks to our assumptions on f , as soon as r is large enough (fixed) and N is larger than a constant times r , we may assume that $\Lambda_M(x) \subset D_N$ for each $x \in A$. Under such conditions Lemma 5.11 tells us that, a.s.,

$$\begin{aligned} E \left(e^{-f(x/N, h^{D_N}(x) - m_N, h^{D_N}(x) - h^{D_N}(x+\cdot) + \Phi^{M,x}(x+\cdot))} 1_{\{x \in \widehat{\Theta}_{N,r}^D\}} \middle| \mathcal{F}_{M,x} \right) \\ = E_{\mathbf{v}^{(M,t)}} \left(e^{-f(x/N, t, \phi)} \right) \Big|_{t:=h^{D_N}(x) - m_N} E(1_{\{x \in \widehat{\Theta}_{N,r}^D\}} \middle| \mathcal{F}_{M,x}) \end{aligned} \quad (5.74)$$

and by conditioning on $\sigma(\mathcal{F}_{M,x} : x \in A)$ we thus get

$$\begin{aligned} E \left(\prod_{x \in A} \left[(e^{-f(x/N, h^{D_N}(x) - m_N, h^{D_N}(x) - h^{D_N}(x+\cdot) + \Phi^{M,x}(x+\cdot))} - 1) 1_{\{x \in \widehat{\Theta}_{N,r}^D\}} \right] \right) \\ = E \left(\prod_{x \in A} \left[(e^{-\tilde{f}_{N,r}(x/N, h^{D_N}(x) - m_N)} - 1) 1_{\{x \in \widehat{\Theta}_{N,r}^D\}} \right] \right). \end{aligned} \quad (5.75)$$

Plugging this back into the inclusion-exclusion formula (5.73), the claim follows. \square

Another reason why the auxiliary process is useful to work with is seen from:

Lemma 5.13 *Fix any $r \geq 1$, any $j \geq 1$ and any $c_1 \in (0, \infty)$. For $M = M(N, r)$ as above, uniformly in $f \in C_b(\mathbb{R}^{\Delta^j})$ with $\|f\|_\infty \leq c_1$ and uniformly in t on compact sets in \mathbb{R} ,*

$$E_{\mathbf{v}^{(M,t)}}(f) \xrightarrow{N \rightarrow \infty} E_{\mathbf{v}}(f), \quad (5.76)$$

where \mathbf{v} is the measure from Proposition 5.8.

Proof. Let n be such that $M = 2^n$ and let, as before, $\mathbf{m}_n(x) = (m_N + t)(1 - \mathfrak{g}^{\Delta^n}(x))$. From Proposition 5.2 and $h^{\Delta^n} \leftrightarrow -h^{\Delta^n}$ symmetry we get

$$E_{\mathbf{v}^{(M,t)}}(f) = \frac{E(f(\mathbf{m}_n - h^{\Delta^n}) 1_{\{h^{\Delta^n} \leq \mathbf{m}_n\}} | h^{\Delta^n}(0) = 0)}{E(1_{\{h^{\Delta^n} \leq \mathbf{m}_n\}} | h^{\Delta^n}(0) = 0)} = \frac{\Xi_\ell^{\text{in}}(f) \Xi_{n,\ell}^{\text{out}}(t) + \tilde{\varepsilon}_n(\ell)}{\Xi_\ell^{\text{in}}(1) \Xi_{n,\ell}^{\text{out}}(t) + \varepsilon_n(\ell)}, \quad (5.77)$$

where $\varepsilon_n(\ell)$ and $\tilde{\varepsilon}_n(\ell)$ tend to zero as $n \rightarrow \infty$ followed by $\ell \rightarrow \infty$, uniformly on compact sets of $t \in \mathbb{R}$ and for f as above. In light of the bounds in (5.45), the right-hand side tends to that of (5.49) which by Proposition 5.8 equals $E_{\mathbf{v}}(f)$. \square

We are now finally ready to give the proof of the first main result of this paper:

Proof of Theorem 2.1. Let $f: D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \rightarrow [0, \infty)$ be a function $f = f(x, h, \phi)$ that is continuous with compact support and dependent only on a finite number of coordinates of ϕ . Consider the expectation $E(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, f \rangle})$ which, thanks to Lemma 5.12, we can replace by $E(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, \tilde{f}_{N,r} \rangle})$ once $N \gg r \gg 1$. Our aim is to derive a limit for this expectation using the results of Biskup and Louidor [11], but for that we will need to replace $\tilde{f}_{N,r}$ by a continuous, compactly-supported function that does not depend on N and r .

For concreteness suppose that $f(x, h, \phi)$ is zero unless $|h| \leq \lambda$ and that f depends only on $\{\phi(x) : x \in \Lambda_{r_0}(0)\}$ for some $r_0 \geq 1$. Define

$$e^{-\tilde{f}_{N,r}(x,h)} = E_{\mathbf{v}(M,h)}(e^{-f(x,h,\phi)}), \quad (5.78)$$

where, as before, $n \in \mathbb{N}$ is such that $M = M(N, r) = 2^n$. By our assumptions on f , the functions $\{\tilde{f}_{N,r} : M(N, r) \geq 1\}$ are all supported in the same compact set in $D \times \mathbb{R}$. Lemma 5.13 then gives that, for each $r \geq 1$,

$$\tilde{f}_{N,r}(x, h) \xrightarrow{N \rightarrow \infty} \tilde{f}(x, h), \quad \text{uniformly in } x \text{ and } h, \quad (5.79)$$

where \tilde{f} is defined by

$$e^{-\tilde{f}(x,h)} = E_{\mathbf{v}}(e^{-f(x,h,\phi)}). \quad (5.80)$$

Both $\tilde{f}_{N,r}$ and \tilde{f} vanishes unless $|h| \leq \lambda$ and so

$$\left| \langle \hat{\eta}_{N,M(N,r)}^D, (\tilde{f} - \tilde{f}_{N,r}) \rangle \right| \leq 2 \|\tilde{f} - \tilde{f}_{N,r}\|_{\infty} |\Gamma_N^D(\lambda)| \quad (5.81)$$

where the random variables $\{|\Gamma_N^D(\lambda)| : N \geq 1\}$ are known to be tight thanks to Lemma B.9. Using this in conjunction with Lemma 5.9, Proposition 5.10 and Lemma 5.12 yields

$$\begin{aligned} E(e^{-\langle \eta_{N,r_N}^D, f \rangle}) + o(1) &= E(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, f \rangle}) \\ &= E(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, \tilde{f}_{N,r} \rangle}) \\ &= E(e^{-\langle \hat{\eta}_{N,M(N,r)}^D, \tilde{f} \rangle}) + o(1) \\ &= E(e^{-\langle \eta_{N,r_N}^D, \tilde{f} \rangle}) + o(1), \end{aligned} \quad (5.82)$$

where the $o(1)$ terms tend to zero in the limits $N \rightarrow \infty$ followed by $r \rightarrow \infty$.

Since $\tilde{f}: D \times \mathbb{R} \rightarrow [0, \infty)$ is continuous with compact support, Theorem 2.1 of Biskup and Louidor [11] shows

$$E(e^{-\langle \eta_{N,r_N}^D, \tilde{f} \rangle}) \xrightarrow{N \rightarrow \infty} E\left(\exp\left\{-\int Z^D(dx) \otimes e^{-\alpha h} dh (1 - e^{-\tilde{f}(x,h)})\right\}\right). \quad (5.83)$$

Invoking the definition of \tilde{f} , the integral in (5.83) can be recast as

$$\int Z^D(dx) \otimes e^{-\alpha h} dh \otimes \mathbf{v}(d\phi) (1 - e^{-f(x,h,\phi)}). \quad (5.84)$$

This yields the desired expression

$$E(e^{-\langle \eta_{N,r_N}^D, f \rangle}) \xrightarrow{N \rightarrow \infty} E\left(\exp\left\{-\int Z^D(dx) \otimes e^{-\alpha h} dh \otimes \mathbf{v}(d\phi) (1 - e^{-f(x,h,\phi)})\right\}\right) \quad (5.85)$$

whenever f is as assumed above. But the class of such f is dense in the class of all continuous compactly-supported functions $f: D \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \rightarrow [0, \infty)$ with respect to the supremum norm, and

so the claim follows by invoking a version of the estimate (5.81) (on the left-hand side) and the fact that $Z^D(D) < \infty$ a.s. (on the right-hand side). \square

Proof of Theorem 2.3. Let ν be the measure constructed by the limit in Proposition 5.8. The proof of Theorem 2.1 (specifically, Lemma 5.13) then shows that ν is indeed the cluster law. \square

Proof of Corollary 2.2. Let $f: \bar{D} \times \mathbb{R} \rightarrow [0, \infty)$ be a continuous function with compact support. For $r \geq 1$ define $f_r: \bar{D} \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \rightarrow [0, \infty)$ by

$$f_r(x, h, \phi) := \sum_{z \in \Lambda_r(0)} f(x, h - \phi_z). \quad (5.86)$$

Let $\lambda > 0$ be such that $f(x, h) = 0$ unless $|h| \leq \lambda$. We then observe that, if $r_N > 2r$, on the event when $x, y \in \Gamma_N^D(\lambda)$ imply either $|x - y| < r$ or $|x - y| > r_N$ (and assuming that no two values of h^{D_N} are the same) each point contributing to $\langle \eta_N^D, f \rangle$ lies within r -neighborhood of a unique r -local maximum of h^{D_N} . Under such circumstances we have

$$\langle \eta_N^D, f \rangle = \sum_{x \in D_N} \sum_{z \in \Lambda_r(x)} \mathbf{1}_{\{h^{D_N}(x) = \max_{z \in \Lambda_r(x)} h^{D_N}(z)\}} f\left(x/N, h^{D_N}(x) - (h^{D_N}(x) - h^{D_N}(z))\right) \quad (5.87)$$

But on the same event we can freely replace $\Lambda_r(x)$ in the indicator on the right-hand side by $\Lambda_{r_N}(x)$ thus leading to $\langle \eta_N^D, f \rangle = \langle \eta_{N, r_N}^D, \tilde{f}_r \rangle$. Using Lemma B.11, it then follows that

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P(\langle \eta_N^D, f \rangle \neq \langle \eta_{N, r_N}^D, \tilde{f}_r \rangle) = 0. \quad (5.88)$$

As the maximum of $h^{D_N} - m_N$ is tight, Theorem 2.1 can be applied for the test function \tilde{f}_r despite the fact that it does not have compact support. Invoking also the Monotone Convergence Theorem to deal with the limit $r \rightarrow \infty$, we get

$$E(e^{-\langle \eta_N^D, f \rangle}) \xrightarrow{N \rightarrow \infty} E\left(\exp\left\{-\int Z^D(dx) \otimes e^{-ah} dh \otimes \nu(d\phi) (1 - e^{-\tilde{f}(x, h, \phi)})\right\}\right), \quad (5.89)$$

where

$$\tilde{f}(x, h, \phi) := \lim_{r \rightarrow \infty} \tilde{f}_r(x, h, \phi) = \sum_{z \in \mathbb{Z}^2} f(x, h - \phi_z). \quad (5.90)$$

The tightness of η_N^D -processes — or the growth estimate on samples from ν in (5.50) — ensure that \tilde{f} is finite almost everywhere under the intensity measure. Since,

$$\int \nu(d\phi) (1 - e^{-\tilde{f}(x, h, \phi)}) = 1 - E_\nu \exp\left\{-\sum_{z \in \mathbb{Z}^2} f(x, h - \phi_z)\right\} \quad (5.91)$$

we now easily check that the right-hand side of (5.89) is the Laplace transform of the cluster process in the statement and that this process is locally finite as claimed. \square

5.4 Control of auxiliary process.

In order to prove Proposition 5.10, we need additional lemmas. The first one shows that the field $\Phi^{M, x}$ is small uniformly in x and its argument.

Lemma 5.14 *For $M = M(N, r)$ as above and any $\delta > 0$,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P\left(\max_{x \in D_N^\delta} \max_{z \in \Lambda_r(x)} |\Phi^{M, x}(z)| > \frac{(\log r)^2}{\sqrt{\log N}}\right) = 0 \quad (5.92)$$

Proof. Fix $\delta > 0$ and assume that N and r are so large that $\Lambda_M(x) \subset D_N^{\delta/2}$ for each $x \in D_N^\delta$. Let $H(x, y)$ abbreviate the harmonic measure $H^{\Lambda_M(0) \setminus \{0\}}(x, y)$. Then

$$\text{Var}(\Phi^{M,x}(x+z)) = \sum_{y, \tilde{y} \in \partial\Lambda_M(0)} H(z, y) H(z, \tilde{y}) G^{D_N}(x+y, x+\tilde{y}). \quad (5.93)$$

Plugging the asymptotic $G^{D_N}(x+y, x+\tilde{y}) \leq c \log(N/(1+|y-\tilde{y}|))$ (cf Lemmas B.3–B.4) implied by the containment $\Lambda_M(x) \subset D_N^\delta$ and using the standard bound on the harmonic measure (cf Lemma B.5)

$$H(z, y) \leq c_1 \frac{\log r}{M \log M}, \quad z \in \Lambda_r(0), y \in \partial\Lambda_M(0), \quad (5.94)$$

where we assume, e.g., $r \leq M/2$, we thus get that, for some constant $c_2 \in (0, \infty)$,

$$\max_{x \in D_N^\delta} \max_{z \in \Lambda_r(x)} |\text{Var}(\Phi(x+z))| \leq c_2 \left(\frac{\log r}{\log M} \right)^2 \log(N/M). \quad (5.95)$$

Since N/M is of order r , the union bound combined with exponential Chebyshev inequality now readily yield the claim. \square

The next lemma uses the above control to show that the nearly-maximal M -local maxima of h^{D_N} exhaust, with high probability, those of $h^{D_N} - \Phi^{M,x}$.

Lemma 5.15 *For any $\lambda > 0$ and $M = M(N, r)$ as above,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P \left(\exists x \in \Gamma_N^D(\lambda) : \begin{array}{l} h^{D_N} - \Phi^{M,x} \leq h^{D_N}(x) \text{ in } \Lambda_M(x) \\ h^{D_N} \not\leq h^{D_N}(x) \text{ in } \Lambda_M(x) \end{array} \right) = 0. \quad (5.96)$$

Proof. Note that, on the event whose probability we are to estimate, there are points $x \in D_N$ and $y \in \Lambda_M(x)$ such that $h^{D_N}(x) \geq m_N - \lambda$ and $h^{D_N}(y) > h^{D_N}(x) \geq h^{D_N}(y) - \Phi^{M,x}(y)$. Unless these points obey $r < |x-y| \leq M$, this y must in fact lie in $\Lambda_r(x)$. Since also $h^{D_N} - \Phi^{M,x} \leq h^{D_N}(x)$ is assumed in $\Lambda_M(x)$, the field $h^{D_N} - \Phi^{M,x}$ then has an r -local maximum at x but with a gap to the next value in $\Lambda_r(x)$ less than $\max_{z \in \Lambda_r(x)} |\Phi^{M,x}(z)|$. Utilizing our uniform bound on this maximum from Lemma 5.14, we will show this to be unlikely to happen anywhere in D_N .

Let $\lambda > 0$ be fixed, pick $\delta > 0$ small and assume that N is so large that, for a given $r \geq 1$, we have $\delta N \gg r$. Abbreviate $a_N := \log \log N / \log N$. Using the above observations, we bound the probability in the statement by

$$\begin{aligned} & P(\Gamma_N^D(\lambda) \setminus D_N^\delta \neq \emptyset) + P\left(\max_{x \in D_N} h^{D_N}(x) > m_N + \lambda'\right) \\ & + P(\exists x, y \in \Gamma_N(\lambda) : r < |x-y| \leq M) \\ & + P\left(\max_{x \in D_N^\delta} \max_{y \in \Lambda_r(x)} |\Phi^{M,x}(y)| > a_N\right) \\ & + \sum_{x \in D_N^\delta} P \left(\begin{array}{l} -\lambda \leq h^{D_N}(x) - m_N \leq \lambda' \\ h^{D_N} - \Phi^{M,x} \leq h^{D_N}(x) \text{ in } \Lambda_M(x) \\ \max_{\substack{y \in \Lambda_r(x) \\ y \neq x}} [h^{D_N}(y) - \Phi^{M,x}(y)] > h^{D_N}(x) - a_N \end{array} \right), \end{aligned} \quad (5.97)$$

where $\lambda' > 0$ and where we used the union bound in the last step. Invoking Lemmas B.12, B.13, B.11 and 5.14, the first four probabilities tend to zero as $N \rightarrow \infty$, $r \rightarrow \infty$ and $\lambda' \rightarrow \infty$. Thanks to

Lemma 5.11, the last probability (without the sum) is equal to

$$\int_{m_N - \lambda}^{m_N + \lambda'} P(h^{D_N}(x) \in dt) P\left(h^{\Delta^n} \leq t \text{ in } \Delta^n, h^{\Delta^n} \not\leq t - a_N \text{ in } \Delta^k \setminus \{0\} \mid h^{\Delta^n}(0) = t\right), \quad (5.98)$$

where $n \in \mathbb{N}$ is such that $M = 2^n$ and $k \in \mathbb{N}$ is such that $2^k \geq r > 2^{k-1}$. Noting that $m_N - m_{2^n}$ remains bounded as $N \rightarrow \infty$ and using Lemma 3.2 to convert the conditioning to $h^{\Delta^n}(0) = 0$, the second part of Lemma 4.22 shows that the integral is bounded by

$$\frac{c_N}{n} P(h^{D_N}(x) \geq m_N - \lambda), \quad (5.99)$$

where $c_N \rightarrow 0$ as $N \rightarrow \infty$. For $x \in D_N^\delta$ we have $\text{Var}(h^{D_N}(x)) \geq g \log N - c$ for some constant $c \in (0, \infty)$. Plugging this to the standard Gaussian asymptotic and using some straightforward manipulations (cf, e.g., (6.34)), the expression (5.99) is bounded by a constant times c_N/N^2 once N is sufficiently large. As $|D_N^\delta|$ is at most a constant times N^2 , the sum in (5.97) tends to zero as $N \rightarrow \infty$, thus proving the claim. \square

The next lemma complements this by showing that the correspondence between the local maxima of h^{D_N} and those of $h^{D_N} - \Phi^{M,x}$ is, in fact, one-to-one with high probability.

Lemma 5.16 *For any $\lambda > 0$ and $M = M(N, r)$ as above,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P\left(\exists x \in \Gamma_N^D(\lambda): \begin{array}{l} h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x) \\ h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x) \text{ in } \Lambda_M(x) \end{array}\right) = 0. \quad (5.100)$$

Proof. Given any $\delta > 0$ and $\lambda' > 0$, by Lemma B.12, we may restrict the event to $x \in D_N^\delta$ and $h^{D_N}(x) \leq m_N + \lambda'$ as soon as N is sufficiently large. We will deal separately with the cases when $h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x)$ occurs in $\Lambda_{M/2}(x)$ and when this occurs in $\Lambda_M(x) \setminus \Lambda_{M/2}(x)$.

CASE 1: Suppose that $h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x)$ occurs in $\Lambda_{M/2}(x)$ for some $x \in D_N^\delta$ and note that, since $h^{D_N} \leq h^{D_N}(x)$, this in particular forces $\Phi^{M,x} \not\geq 0$ in $\Lambda_{M/2}(x)$. Applying the union bound, the relevant probability is at most

$$\sum_{x \in D_N^\delta} P\left(h^{D_N}(x) - m_N \in [-\lambda, \lambda'], h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x), \Phi^{M,x} \not\geq 0 \text{ in } \Lambda_{M/2}(x)\right). \quad (5.101)$$

We will bound the probability under the sum by conditioning on $h^{D_N}(x) = m_N + t$ (with $t \in [-\lambda, \lambda']$) and on $h^{D_N} \leq m_N + t$ in $\Lambda_M(x)$.

Focusing on the conditional probability only, the strong-FKG property (Lemma B.8) and the fact that $\Phi^{M,x}$ is an increasing function of h^{D_N} then imply

$$\begin{aligned} & P\left(\Phi^{M,x} \not\geq 0 \text{ in } \Lambda_{M/2}(x) \mid h^{D_N} \leq m_N + t \text{ in } \Lambda_M(x), h^{D_N}(x) = m_N + t\right) \\ & \leq P\left(\Phi^{M,x} \not\geq 0 \text{ in } \Lambda_{M/2}(x) \mid h^{D_N} \leq m_N + t \text{ in } D_N, h^{D_N}(x) = m_N + t\right). \end{aligned} \quad (5.102)$$

We now assume for simplicity (and without loss of generality) that $x = 0$ and, abusing our earlier notation, interpret D_N as a domain such that

$$\Lambda_M(0) = \Delta^n \subseteq D_N =: \Delta^{n+q}, \quad (5.103)$$

where $n \in \mathbb{N}$ is such that $M = 2^n$ and where $q = q(r) = r \log 2 + O(1)$. In this notation, Lemma 3.2 shows that the conditional probability becomes

$$P\left(\Phi^{M,0} + (m_N + t)\mathfrak{g}^{D_N} \not\geq 0 \text{ in } \Delta^{n-1} \mid h^{D_N} \leq (m_N + t)(1 - \mathfrak{g}^{D_N}), h^{D_N}(x) = 0\right) \quad (5.104)$$

where $\Phi^{M,0}$ admits the representation

$$\Phi^{M,0}(z) = \sum_{k=n+1}^{n+q} (\mathfrak{b}_k(z)\phi_k(0) + \chi_k(z)), \quad z \in \Delta^n(0), \quad (5.105)$$

which is checked by noting that the field on the right agrees with h^{D_N} on $\partial\Delta^n$, is harmonic in $\Delta^n \setminus \{0\}$ and equal to zero at $z = 0$. In light of the harmonicity of $z \mapsto \Phi^{M,0}(z) + (m_N + t)\mathfrak{g}^{D_N}(z)$ on $D_N \setminus \{0\}$ and the fact that this function vanishes at $z := 0$, by the maximum principle it suffices to estimate the probability that this function takes a negative value on $\partial\Delta^{n-1}$. For that we note

$$m_N \mathfrak{g}^{D_N}(z) = 2\sqrt{g} \log r + O(1), \quad z \in \partial\Delta^{n-1}, \quad (5.106)$$

and so we can bound the probability in (5.104) by

$$P\left(\Phi^{M,0} \not\geq -2\sqrt{g} \log r - c \text{ on } \partial\Delta^{n-1} \mid h^{D_N} \leq (m_N + t)(1 - \mathfrak{g}^{D_N}), h^{D_N}(x) = 0\right) \quad (5.107)$$

for some $c = c(\lambda) \in (0, \infty)$.

We are now ready to derive the desired bound. Comparing the event in (5.107) with (5.105) and assuming r to be large, on the event in question we have

$$\max\left\{\max_{n \leq k \leq n+q} |\phi_k(0)|, \max_{n \leq k \leq n+q} \max_{z \in \Delta^n} |\chi_k(z)|\right\} > c' \log r \quad (5.108)$$

for some $c' \in (0, \infty)$ depending only on the constant in Lemma 3.8. As was used in the proof of Lemma 4.2, the conditioning on $h^{D_N}(0) = 0$ amounts to changing $\phi_k(0)$ to $\phi_k(0) - c_n(k)S_{n+1}$ which have a uniform Gaussian tail under the conditioning. By Lemma 3.8 a similar statement holds for the χ_k 's. Following the proof of Lemmas 4.20, and invoking Lemma 4.21 for the lower bound on the conditional event, the probability in (5.107) is thus at most $c_1 e^{-c_2(\log r)^2}$ uniformly in $t \in [-\lambda, \lambda']$ (and uniformly in all shifts of D_N for which 0 lies in D_N^δ).

The quantity in (5.101) is therefore bounded by

$$c_1 e^{-c_2(\log r)^2} \sum_{x \in D_N^\delta} P\left(x \in \Gamma_N^D(\lambda), h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x)\right). \quad (5.109)$$

Writing the sum as an expectation of the sum of indicators, the fact that the field values are continuously distributed and $M \geq N/(2r)$ shows that the sum (of indicators) is bounded pointwise (a.s.) by a constant times r^2 , uniformly in N (sufficiently large). The expression in (5.109) thus tends to zero as $r \rightarrow \infty$, thus taking care of the case when $h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x)$ occurs in $\Lambda_{M/2}(x)$.

CASE 2: Let us now assume that $h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x)$ occurs in $A_M(x) := \Lambda_M(x) \setminus \Lambda_{M/2}(x)$. By way of a union bound, we can fix that x and estimate the probability for that x only. Recall the notation $\mathcal{F}_{M,x}$ for the σ -algebra from Lemma 5.11. Noting that $\Phi^{M,x}$ and $h^{D_N}(x)$ are measurable

with respect to $\mathcal{F}_{M,x}$, by conditional independence (cf Lemma 5.11)

$$\begin{aligned} P\left(h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x), h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x) \text{ in } A_M(x) \mid \mathcal{F}_{M,x}\right) \\ = P\left(h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x) \mid \mathcal{F}_{M,x}\right) \\ \times P\left(h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x) \text{ in } A_M(x) \mid \mathcal{F}_{M,x}\right). \end{aligned} \quad (5.110)$$

Letting $n \in \mathbb{N}$ be such that $M = 2^n$, Lemma 5.11 along with the fact that $(m_N - \lambda)(1 - \mathfrak{g}^{\Delta^n}) - m_M \geq 2\sqrt{g} \log r - c$ on $\Delta^n \setminus \Delta^{n-1}$ then shows that on $\{h^{D_N}(x) \geq m_N - \lambda\}$ we have the pointwise (a.s.) inequality

$$\begin{aligned} P\left(h^{D_N} - \Phi^{M,x} \not\leq h^{D_N}(x) \text{ in } A_M(x) \mid \mathcal{F}_{M,x}\right) \\ = P\left(h^{\Delta^n \setminus \{0\}} - (m_N + t)(1 - \mathfrak{g}^{\Delta^n}) \not\leq 0 \text{ in } \Delta^n \setminus \Delta^{n-1} \mid t := h^{D_N}(x)\right) \\ \leq P\left(\max_{z \in \Delta^n \setminus \Delta^{n-1}} h^{\Delta^n \setminus \{0\}}(z) > m_M + 2\sqrt{g} \log r - c\right). \end{aligned} \quad (5.111)$$

Using Lemma B.7 to replace $h^{\Delta^n \setminus \{0\}}$ by h^{Δ^n} , the sharp upper tail of the maximum in Lemma B.12 bounds this probability by a constant times $r^{-4} \log r$. Using this in (5.110), the probability in the case under consideration is then bounded by a constant times

$$\frac{\log r}{r^4} \sum_{x \in D_N^\delta} P(h^{D_N} \leq h^{D_N}(x) \text{ in } \Lambda_M(x)). \quad (5.112)$$

As argued before, the sum is at most a constant times r^2 uniformly in N and so the claim follows by taking $r \rightarrow \infty$. \square

With the above lemmas in hand, the proof of the last missing step in the proof of our main theorems is now quite immediate:

Proof of Proposition 5.10. We will write M in place of $M(N, r)$. Suppose that $f(x, h, \phi)$ is zero unless $|h| \leq \lambda$ and assume f only depends on $\phi(z)$ for $z \in \Lambda_{r_0}(0)$ for some $r_0 \geq 1$. Introduce an intermediate (auxiliary) process

$$\tilde{\eta}_{N,M}^D := \sum_{x \in D_N} 1_{\{h_x^{D_N} = \max_{z \in \Lambda_M(x)} h_z^{D_N}\}} \delta_{x/N} \otimes \delta_{h_x^{D_N} - m_N} \otimes \delta_{\{h_x^{D_N} - h_{x+z}^{D_N} + \Phi_{x+z}^{M,x} : z \in \mathbb{Z}^2\}}. \quad (5.113)$$

Lemmas 5.15–5.16 show that the M -local extrema of h^{D_N} and $h^{D_N} - \Phi^{M,\cdot}$ are in one-to-one correspondence with probability tending to one in the limit as $N \rightarrow \infty$ and $r \rightarrow \infty$. It follows that, for any f as above,

$$\lim_{r \rightarrow \infty} \lim_{N \rightarrow \infty} P(\langle \tilde{\eta}_{N,M}^D, f \rangle \neq \langle \hat{\eta}_{N,M}^D, f \rangle) = 0. \quad (5.114)$$

But the assumptions on continuity and support of f imply that for each $\varepsilon > 0$ there is $\delta > 0$ such that for all x and h ,

$$\max_{x \in \Lambda_r(0)} |\phi(x) - \phi'(x)| < \delta \quad \Rightarrow \quad |f(x, h, \phi) - f(x, h, \phi')| < \varepsilon. \quad (5.115)$$

In light of Lemma 5.14, for each $\varepsilon > 0$ we thus have

$$\lim_{N \rightarrow \infty} P(|\langle \tilde{\eta}_{N,M}^D, f \rangle - \langle \eta_{N,M}^D, f \rangle| > \varepsilon) = 0 \quad (5.116)$$

as soon as $r > r_0$. The claim follows. \square

6. LOCAL LIMIT THEOREM AND FREEZING

In this section we complete the proofs of the remaining results: the local limit theorem for the absolute maximum (Theorem 2.5) and then the results on the Liouville measure in glassy phase and freezing (Theorem 2.6 and Corollaries 2.7–2.8).

6.1 Local limit theorem for the absolute maximum.

We begin with the local limit theorem for both position and value of the global maximum. The starting point is a reformulation of Proposition 5.2 for general outer domains. For any integer $q \in \mathbb{Z}$ define the square domain

$$S^q := (-2^q, 2^q) \times (-2^q, 2^q). \quad (6.1)$$

Thanks to translation invariance of the DGFF and scaling, we can and will restrict attention to continuum domains in the class

$$\mathfrak{D}_q := \{D \in \mathfrak{D} : \overline{S^1} \subseteq D \subseteq S^q\} \quad (6.2)$$

where $q > 1$ is an arbitrary (but fixed) integer. The definition ensures the following property: If $\{D_N\}$ is a sequence of approximating domains satisfying (2.1–2.2) and

$$n := \min\{m \in \mathbb{N} : 2^m \geq N\}, \quad (6.3)$$

then for all N sufficiently large,

$$\Delta^{n-\ell-1} \subseteq S_N^{-\ell} \subseteq \Delta^{n-\ell} \subseteq \Delta^n \subseteq D_N \subseteq \Delta^{n+q}, \quad (6.4)$$

where we set $S_N^{-\ell} := \{x \in \mathbb{Z}^2 : x/N \in S^{-\ell}\}$. The reason for inserting $S_N^{-\ell}$ in-between $\Delta^{n-\ell-1}$ and $\Delta^{n-\ell}$ is that, unlike these two domains (assuming n is tied to N as in (6.3)), $S_N^{-\ell}$ has a well defined scaling limit as $N \rightarrow \infty$.

Pick $\ell \geq 1$ and given $D \in \mathfrak{D}_q$, consider a sequence $\{D_N\}$ of approximating domains satisfying (2.1–2.2) and (6.4). For h^{D_N} the DGFF in D_N , set

$$\Psi_{N,\ell}(x) := E(h^{D_N}(x) \mid \sigma(h^{D_N}(z) : z \in \partial S_N^{-\ell})). \quad (6.5)$$

For each $t \in \mathbb{R}$ and for m_N as in (1.2), we then define an analogue of $\Xi_{n,\ell}^{\text{out}}(t)$ from (5.3) as

$$\Xi_{N,\ell}^D(t) := E\left(\Psi_{N,\ell}(0) \mathbf{1}_{\{\Psi_{N,\ell}(0) \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in D_N \setminus S_N^{-\ell+1}} \mathbf{1}_{\{h^{D_N}(x) \leq m_N(x,t)\}} \mid h^{D_N}(0) = 0\right), \quad (6.6)$$

where, abusing our earlier notation slightly,

$$m_N(x,t) := (m_N + t)(1 - \mathfrak{g}^{D_N}(x)) \quad (6.7)$$

with \mathfrak{g}^D as defined in Section 3.1. Recall that $\Xi_\ell^{\text{in}}(1)$ denotes the quantity from (5.1) for $f := 1$ and that $\Xi_\infty^{\text{in}}(1) = \lim_{\ell \rightarrow \infty} \Xi_\ell^{\text{in}}(1)$ exists. Our rewrite of Proposition 5.2 is now as follows:

Proposition 6.1 *For each $q > 1$, each $\varepsilon > 0$ and each $t_0 > 0$ there is $\ell_0 \geq 1$ such that for all $t \in \mathbb{R}$ with $|t| < t_0$, all $\ell \geq \ell_0$ and all $D \in \mathfrak{D}_q$,*

$$\left| P(h^{D_N} \leq m_N + t \mid h^{D_N}(0) = m_N + t) - \frac{2}{g \log N} \Xi_\infty^{\text{in}}(1) \Xi_{N,\ell}^D(t) \right| \leq \frac{\varepsilon}{\log N} \quad (6.8)$$

holds true for any sequence $\{D_N\}$ corresponding to D via (2.1–2.2) as soon as N is so large that $\ell \leq (\log N)^{1/8}$ and (6.4) apply.

Proof. Applying the observations from Lemma 4.23, the proof of Proposition 5.2 carries over to this case as soon as N and ℓ are such that $\ell \leq (\log N)^{1/8}$ and (6.4) hold. Recalling (6.3) and denoting, for $a > 0$,

$$\tilde{\Xi}_{N,\ell}^D(t, a) := E \left(S_{n-\ell} 1_{\{S_{n-\ell} \in [a^{-1}\ell^{1/6}, a\ell^2]\}} \prod_{x \in D_N \setminus \Delta^{n-\ell}} 1_{\{h^{D_N}(x) \leq m_N(x,t)\}} \middle| S_{n+1} = 0 \right) \quad (6.9)$$

where the random walk $\{S_k\}$ is now related to h^{D_N} via $S_{n+1} = h^{D_N}(0)$ and

$$S_k = E(h^{D_N}(0) \mid \sigma(h^{D_N}(z) : z \in \partial\Delta^{k-1})), \quad k = 1, \dots, n, \quad (6.10)$$

we thus get

$$\begin{aligned} & \frac{2}{g \log 2} \frac{1}{n} \Xi_\ell^{\text{in}}(1) \tilde{\Xi}_{N,\ell}^D(t, 1/2) - \frac{\varepsilon}{n} \\ & \leq P(h^{D_N} \leq m_N + t \mid h^{D_N}(0) = m_N + t) \\ & \leq \frac{2}{g \log 2} \frac{1}{n} \Xi_{\ell+1}^{\text{in}}(1) \tilde{\Xi}_{N,\ell+1}^D(t, 2) + \frac{\varepsilon}{n}. \end{aligned} \quad (6.11)$$

Now the inclusions $\Delta^{n-\ell-1} \subseteq S_N^{-\ell} \subseteq \Delta^{n-\ell}$ from (6.4) and the Gibbs-Markov property (Lemma B.6) ensure that $S_k - \Psi_{N,\ell}(0)$ is a centered Gaussian with a uniformly bounded variance. A straightforward estimate (of the kind done in Lemma 4.2) then shows that the contribution to $\tilde{\Xi}_{N,\ell}^D(t, a)$ of the event when this random variable is larger than $\frac{1}{2}\ell^{1/6}$ is at most ε as soon as ℓ is large enough (we use that $S_{n-\ell}$ is bounded by $2\ell^2$ when $a \leq 2$). This yields

$$\tilde{\Xi}_{N,\ell+1}^D(t, 2) - \varepsilon \leq \Xi_{N,\ell}^D(t) \leq \tilde{\Xi}_{N,\ell}^D(t, 1/2) + \varepsilon. \quad (6.12)$$

Invoking $n \log 2 = (1 + o(1)) \log N$, applying the corresponding analogue of (5.44–5.45) and using that $\Xi_\ell^{\text{in}}(1)$ can be made as close to $\Xi_\infty^{\text{in}}(1)$ by taking ℓ large enough, the claim follows (relabeling ε to a constant times ε to absorb numerical prefactors). \square

We record one consequence of the proof:

Corollary 6.2 *For each $q > 1$ and each $t_0 > 0$ there are $c_1, c_2 \in (0, \infty)$ and $\ell_0 \geq 1$ such that for all $\ell \geq \ell_0$, all $D \in \mathfrak{D}_q$ and any sequence $\{D_N\}$ of lattice domains such that (2.1–2.2) apply,*

$$c_1 < \Xi_{N,\ell}^D(t) < c_2 \quad (6.13)$$

holds true uniformly in $t \in [-t_0, t_0]$ as soon as N is sufficiently large.

Proof. This follows from Proposition 6.1, the fact that $\Xi_\infty^{\text{in}}(1) \in (0, \infty)$ and the bounds in Lemma 4.21 adapted to the present situation. \square

The behavior of $N \mapsto \Xi_{N,\ell}^D(t)$ as $N \rightarrow \infty$ was not important for the construction of the cluster law as this quantity factors out from all relevant formulas. This is different for the local limit theorem for the maximum, where we will need the limit of $\Xi_{N,\ell}^D(t)$ as $N \rightarrow \infty$ to exist and even have some regularity properties. To describe the limit object, let $\hat{\Psi}_\ell$ denote the mean-zero Gaussian process

on $D \setminus \partial S^{-\ell}$ with covariance

$$C(x, y) := G^D(x, y) - G^{S^{-\ell}}(x, y), \quad (6.14)$$

where G^D is the continuum Green function in D with Dirichlet boundary conditions on ∂D ; see Section B.2. As it turns out, the field $\widehat{\Psi}_\ell$ is the scaling limit of $\Psi_{N,\ell}$ defined above:

Lemma 6.3 *Fix $\ell \geq 1$, $q > 1$ and let $D \in \mathfrak{D}_q$. The field $\widehat{\Psi}_\ell$ has continuous sample paths on $D \setminus \partial S^{-\ell}$ a.s. Moreover, for each $N \geq 1$ there is a coupling of $\Psi_{N,\ell}$ and $\widehat{\Psi}_\ell$ such that, for each $\delta > 0$ sufficiently small,*

$$\max_{\substack{x \in D_N \\ \text{dist}(x, \partial D_N \cup \partial S_N^{-\ell}) > \delta N}} |\Psi_{N,\ell}(x) - \widehat{\Psi}_\ell(x/N)| \xrightarrow{N \rightarrow \infty} 0, \quad \text{in probability.} \quad (6.15)$$

Proof (sketch). As $\Psi_{N,\ell}$, resp., $\widehat{\Psi}_\ell$ is the discrete and continuum “binding” field relating the GFF in D to that in $D \setminus \partial S^{-\ell}$, this reduces to Lemma B.14. \square

The claim about $\Xi_{N,\ell}^D(t)$ we will need is then as follows:

Proposition 6.4 *Let $q > 1$ be an integer. For each $\ell \geq 1$, each $t \in \mathbb{R}$, each $D \in \mathfrak{D}_q$ and each sequence $\{D_N\}$ of domains related to D as in (2.1–2.2), the limit*

$$\Xi_{\infty,\ell}^D(t) := \lim_{N \rightarrow \infty} \Xi_{N,\ell}^D(t) \quad (6.16)$$

exists and is finite, strictly positive, non-decreasing and continuous in t . Moreover, the limit is independent of the sequence $\{D_N\}$ and, in fact, admits the explicit representation

$$\Xi_{\infty,\ell}^D(t) = E \left(\widehat{\Psi}_\ell(0) Q_{\ell,t}^D(\widehat{\Psi}_\ell) 1_{\{\widehat{\Psi}_\ell(0) \in [\ell^{1/6}, \ell^2]\}} \right), \quad (6.17)$$

where $Q_{\ell,t}^D(\varphi)$ is, for each Borel measurable $\varphi: \overline{D} \setminus S^{-\ell+1} \rightarrow \mathbb{R}$, given by

$$Q_{\ell,t}^D(\varphi) = E \left(\exp \left\{ -\alpha^{-1} e^{-\alpha t} \int_{D \setminus S^{-\ell+1}} Z^{D \setminus S^{-\ell}}(dx) e^{\alpha \varphi(x) + \alpha^2 G^D(0,x)} \right\} \right) \quad (6.18)$$

Here Z^D is the measure from Theorem 2.1.

Proof. Fix $\ell \geq 1$ and define, for $\alpha := 2/\sqrt{g}$ and G^D the continuum Green function in D ,

$$\widehat{\Xi}_{N,\ell}^D(t) := E \left(\widehat{\Psi}_\ell(0) 1_{\{\widehat{\Psi}_\ell(0) \in [\ell^{1/6}, \ell^2]\}} \prod_{x \in D_N \setminus S_N^{-\ell+1}} 1_{\{h^{D_N \setminus S_N^{-\ell}}(x) + \widehat{\Psi}_\ell(x/N) \leq m_N + t - \alpha G^D(0, x/N)\}} \right) \quad (6.19)$$

where we regard $h^{D_N \setminus S_N^{-\ell}}$ and $\widehat{\Psi}_\ell$ as independent. Abusing our earlier notation, let $\widetilde{\Xi}_{N,\ell}^D(t)$ be the same quantity but with all occurrences of $\widehat{\Psi}_\ell(\cdot/N)$ replaced by $\Psi_{N,\ell}(\cdot)$. Thanks to the Gibbs-Markov property (Lemma B.6), $\widetilde{\Xi}_{N,\ell}^D(t)$ is the quantity defined as in (6.6) but without the conditioning on $h^{D_N}(0) = 0$ and with $m_N(x, t)$ replaced by $m_N + t - \alpha G^D(0, x/N)$.

We claim that the three objects $\Xi_{N,\ell}^D(t)$, $\widetilde{\Xi}_{N,\ell}^D(t)$ and $\widehat{\Xi}_{N,\ell}^D(t)$ are equal in the limit as $N \rightarrow \infty$. Indeed, to see the closeness of the former two note that

$$\lim_{N \rightarrow \infty} \max_{x \in D_N \setminus S_N^{-\ell}} |m_N(x, t) - (m_N + t - \alpha G^D(0, x/N))| = 0 \quad (6.20)$$

uniformly on compact sets of t . In addition, observe that

$$(h^{D_N}(\cdot) | h^{D_N}(0) = 0) \stackrel{\text{law}}{=} h^{D_N}(\cdot) - \mathbf{g}^{D_N}(\cdot) h^{D_N}(0) \quad (6.21)$$

and

$$\max_{x \in D_N \setminus S_N^{-\ell}} \mathbf{g}^{D_N}(x) \leq c \frac{\ell}{\log N} \quad (6.22)$$

for some $c \in (0, \infty)$. Let ε_N be a sequence with $\varepsilon_N \downarrow 0$ such that ε_N is larger than both $c \frac{\ell}{\log N}$ and the maximum in (6.20) and such that the probability that $h^{D_N}(0) \in [-(\log N)^{1/3}, (\log N)^{1/3}]$ or that $\Psi_{N,\ell}(0)$ is within ε_N of $\ell^{1/3}$ or ℓ^2 is at most $1/N$. Then

$$(1 - \varepsilon_N) \tilde{\Xi}_{N,\ell}^D(t - 2\varepsilon_N) - 2\ell^2/N \leq \Xi_{N,\ell}^D(t) \leq (1 + \varepsilon_N) \tilde{\Xi}_{N,\ell}^D(t + 2\varepsilon_N) + 2\ell^2/N. \quad (6.23)$$

and so $\Xi_{N,\ell}^D(t) - \tilde{\Xi}_{N,\ell}^D(t)$ indeed converges to zero as $N \rightarrow \infty$.

Moving to the corresponding relation with $\hat{\Xi}_{N,\ell}^D(t)$, consider the coupling of $\Psi_{N,\ell}$ and $\hat{\Psi}_\ell$ guaranteed by Lemma 6.3 for δ such that $\text{dist}(0, \partial S_N^{-\ell}) > \delta N$ as well as $\text{dist}(\partial S_N^{-\ell+1}, S_N^{-\ell}) > \delta N$. (This ensures that (6.15) applies to all occurrences of $\Psi_{N,\ell}$ in (6.19).) Let $\tilde{\varepsilon}_N$ be a sequence $\tilde{\varepsilon}_N \downarrow 0$ such that with probability at least $1 - 1/N$, the maximum in (6.15) is at most $\tilde{\varepsilon}_N$ and $\hat{\Psi}_\ell(0)$ lies further than $\tilde{\varepsilon}_N$ of the endpoints of the interval $[\ell^{1/6}, \ell^2]$. Then

$$(1 - \tilde{\varepsilon}_N) \hat{\Xi}_{N,\ell}^D(t - 2\tilde{\varepsilon}_N) - 2\ell^2/N \leq \tilde{\Xi}_{N,\ell}^D(t) \leq (1 + \tilde{\varepsilon}_N) \hat{\Xi}_{N,\ell}^D(t + 2\tilde{\varepsilon}_N) + 2\ell^2/N \quad (6.24)$$

and so $\hat{\Xi}_{N,\ell}^D(t) - \tilde{\Xi}_{N,\ell}^D(t)$ also converges to zero as $N \rightarrow \infty$. Combining (6.23–6.24), it suffices to prove the claim for $\hat{\Xi}_{N,\ell}^D$ instead of $\Xi_{N,\ell}^D$.

First we note that the product in (6.19) is the $a \rightarrow \infty$ limit of $e^{-a\langle \eta_N^D, f_{\hat{\Psi}} \rangle}$ where

$$f_{\hat{\Psi}}(x, h) := 1_{[t - \hat{\Psi}_\ell(x) - \alpha G^D(0, x), \infty)}(h) 1_{D \setminus S^{-\ell+1}}(x). \quad (6.25)$$

Given a sample path of $\hat{\Psi}_\ell(x)$, this function can in turn be approximated by continuous functions with compact support in $\bar{D} \times \mathbb{R}$. The full (unstructured) process convergence in Corollary 2.2 and some routine use of the Monotone Convergence Theorem then show that

$$\lim_{N \rightarrow \infty} \hat{\Xi}_{N,\ell}^D(t) = E \left(\hat{\Psi}_\ell(0) Q_{\ell,t}^D(\hat{\Psi}_\ell) 1_{\{\hat{\Psi}_\ell(0) \in [\ell^{1/6}, \ell^2]\}} \right), \quad (6.26)$$

holds for all t , where η^D is the limit point process on the right-hand side of (2.6) and $Q_{\ell,t}^D(\varphi)$ is the probability

$$Q_{\ell,t}^D(\varphi) := P \left(\eta^D \left(\{ (x, h) \in (\bar{D} \setminus S^{-\ell+1}) \times \mathbb{R} : h + \varphi(x) + \alpha G^D(0, x) > t \} \right) = 0 \right), \quad (6.27)$$

and the right-hand side is continuous in t .

Thanks to Theorem 2.1, this probability admits the explicit representation (6.18). The function $t \mapsto Q_{\ell,t}^D(\varphi)$, being essentially a Laplace transform of a non-negative and finite random variable, is automatically continuous, non-vanishing and finite for all $t \in \mathbb{R}$ and all φ as above. In particular, (6.26) applies for all t and, by monotonicity, the convergence is locally uniform. \square

Using a similar argument as in Proposition 5.8, we now get:

Corollary 6.5 *For all $t \in \mathbb{R}$, all integers $q > 1$ and all $D \in \mathfrak{D}_q$, the limit*

$$\Xi_{\infty,\infty}^D(t) := \lim_{\ell \rightarrow \infty} \Xi_{\infty,\ell}^D(t) \quad (6.28)$$

exists and is positive, finite, non-decreasing and continuous in t . Moreover, for any sequence $\{D_N\}$ of domains related to D via (2.1–2.2),

$$\lim_{N \rightarrow \infty} (\log N) P\left(h^{D_N} \leq m_N + t \mid h^{D_N}(0) = m_N + t\right) = (2/g) \Xi_{\infty}^{\text{in}}(1) \Xi_{\infty, \infty}^D(t). \quad (6.29)$$

The sequence on the left is bounded uniformly in N .

Proof. By Propositions 6.1 and 6.4, for each $\varepsilon > 0$ and each $t_0 > 0$, the sequence under the limit on the left of (6.29) is within ε of $(2/g) \Xi_{\infty}^{\text{in}}(1) \Xi_{\infty, \ell}^D(t)$ uniformly in $t \in [-t_0, t_0]$ as soon as $N \gg \ell \gg 1$. As the former of these two sequences does not depend on ℓ , the limit in (6.28) exists and (6.29) holds. Since $t \mapsto \Xi_{\infty, \ell}^D(t)$ is continuous, $t \mapsto \Xi_{\infty, \infty}^D(t)$ is continuous as well. \square

In light of the scaling relations for the Z^D -measure (cf Corollary 2.2 of [11]), the limit (6.28) exists and defines $\Xi_{\infty, \infty}^D(t)$ for any $D \in \mathfrak{D}$ with $0 \in D$. For arbitrary $D \in \mathfrak{D}$ we then set

$$\Xi_{\infty, \infty}^D(t, x) := \Xi_{\infty, \infty}^{-x+D}(t), \quad x \in D. \quad (6.30)$$

Our last item of concern is the regularity of $x \mapsto \Xi_{\infty, \infty}^D(t, x)$:

Lemma 6.6 *Let $D \in \mathfrak{D}$ with $0 \in D$. Then $x \mapsto \Xi_{\infty, \infty}^D(t, x)$ is continuous on D for each $t \in \mathbb{R}$.*

Proof. The argument from the proof of Corollary 6.5 shows that $\Xi_{\infty, \ell}^D(t)$ approximates $\Xi_{\infty, \infty}^D(t)$ uniformly in $D \in \mathfrak{D}_q$. It thus suffices to show that $x \mapsto \Xi_{\infty, \ell}^{-x+D}(t)$ is continuous at $x := 0$ for each ℓ and each $D \in \mathfrak{D}_q$. For this observe that $x \mapsto G^{-x+D}(0, \cdot)$ varies continuously on $D \setminus S^{-\ell+1}$ while both $x \mapsto Z^{-x+D}$ and $x \mapsto \hat{\Psi}_{\ell}^{-x+D}$, where $\hat{\Psi}_{\ell}^D$ marks the explicit dependence of the above field $\hat{\Psi}_{\ell}$ on the underlying domain, are continuous in law for x small. The continuity of $x \mapsto \Xi_{\infty, \ell}^{-x+D}(t)$ is then checked from (6.17–6.18). \square

Proof of Theorem 2.5. Let $D \in \mathfrak{D}$ and suppose without loss of generality that $0 \in D$. We will prove the claim with $\rho^D(t, x)$ given by

$$\rho^D(x, t) := c e^{-\alpha t} \exp\left\{2 \int_{\partial D} \Pi^D(x, dz) \log |x - z|\right\} \Xi_{\infty}^{\text{in}}(1) \Xi_{\infty, \infty}^D(t, x), \quad (6.31)$$

where c is a constant to be determined, $\Pi^D(x, \cdot)$ is the harmonic measure on ∂D for the Brownian motion started from x .

Let $\{D_N\}$ be a sequence of domains related to D via (2.1–2.2) and let $x_N := \lfloor xN \rfloor$. The probability density of $h^{D_N}(x_N)$ evaluated at $m_N + t$ is then

$$f_N(x, t) := \frac{1}{\sqrt{2\pi \text{Var}(h^{D_N}(x_N))}} e^{-\frac{1}{2} \frac{(m_N + t)^2}{\text{Var}(h^{D_N}(x_N))}}. \quad (6.32)$$

The standard representation of the discrete Green function using the potential yields

$$\text{Var}(h^{D_N}(x_N)) = g \log N + g \int_{\partial D} \Pi^D(x, dz) \log |x - z| + c_0 + O(N^{-2}). \quad (6.33)$$

After some straightforward manipulations, this shows

$$\begin{aligned} \frac{1}{2} \frac{(m_N + t)^2}{\text{Var}(h^{D_N}(x_N))} &= 2 \log N - \frac{3}{2} \log \log N \\ &\quad + \alpha t - 2 \int_{\partial D} \Pi^D(x, dz) \log |x - z| - 2c_0/g + o(1), \end{aligned} \quad (6.34)$$

where $o(1) \rightarrow 0$ uniformly on compact sets of t and on compact sets of $x \in D$. Hence we get

$$f_N(x, t) = \frac{e^{2c_0/g+o(1)}}{2} e^{-\alpha t} \exp\left\{2 \int_{\partial D} \Pi^D(x, dz) \log|x-z|\right\} \frac{\log N}{N^2} \quad (6.35)$$

and, using also (6.29) and the translation invariance of the DGFF,

$$\lim_{N \rightarrow \infty} N^2 P\left(h^{D_N} \leq m_N + t \mid h^{D_N}(x_N) = m_N + t\right) f_N(x, t) = \rho^D(x, t) \quad (6.36)$$

provided we set $c := e^{2c_0/g}/g$ in (6.31). Since

$$\begin{aligned} P(h^{D_N} \leq h^{D_N}(z), h^{D_N}(z) - m_N \in (a, b)) \\ = \int_a^b P(h^{D_N} \leq m_N + t \mid h^{D_N}(z) = m_N + t) f_N(z/N, t) dt \end{aligned} \quad (6.37)$$

and since the integrands on the right are bounded uniformly on compact sets of t , the Bounded Convergence Theorem then proves (2.11).

It remains to connect $\rho^D(x, t)$ to the measure (2.12). Pick $A \subset D$ open with $\bar{A} \subset D$ and recall that by Corollary 1.2 of Biskup and Louidor [10] generalized, with the help of Theorem 2.1, to arbitrary domains in \mathfrak{D} ,

$$\begin{aligned} P\left(N^{-1} \operatorname{argmax}_{D_N} h^{D_N} \in A, \max_{z \in D_N} h^{D_N}(z) - m_N \in (a, b)\right) \\ \xrightarrow{N \rightarrow \infty} \int_a^b E(\widehat{Z}(A) e^{-\alpha^{-1} Z^D(D) e^{-\alpha t}}) dt. \end{aligned} \quad (6.38)$$

Writing the left-hand side as

$$\int_{A \times (a, b)} N^2 P(h^{D_N} \leq m_N + t \mid h^{D_N}(x_N) = m_N + t) f_N(x, t) dx dt \quad (6.39)$$

and recalling that, by Corollary 6.5 and (6.35), the integrand is bounded uniformly in N , (6.36), the Bounded Convergence Theorem, the fact that ρ^D is measurable and (6.38) then show

$$\int_{A \times (a, b)} \rho^D(x, t) dx dt = \int_a^b E(\widehat{Z}(A) e^{-\alpha^{-1} Z^D(D) e^{-\alpha t}}) dt. \quad (6.40)$$

As this holds for a generating class of sets A , the continuity of ρ^D yields the desired claim. \square

Remark 6.7 Recall that, for each $D \in \mathfrak{D}$, the function ψ^D takes the form

$$\psi^D(x) = c_* \exp\left\{2 \int_{\partial D} \Pi^D(x, dz) \log|x-z|\right\} \quad (6.41)$$

for some (existential) constant $c_* \in (0, \infty)$ (same as that in (2.14)); see Biskup and Louidor [11]. Comparing (6.31) with (2.13) and recalling the notation $c := e^{2c_0/g}/g$ from the above proof, it thus follows that

$$c_* = c \Xi_\infty^{\text{in}}(1) \lim_{t \rightarrow \infty} \frac{\Xi_{\infty, \infty}^D(t, x)}{t}, \quad (6.42)$$

where, in particular, the limit exists and is independent of x . The limit depends only on the global characteristics of the extreme value statistics (as expressed by the Z^D measure); all local properties are encoded into $\Xi_\infty^{\text{in}}(1)$ and, to some extent, also the constant c (which depends on the potential \mathfrak{a} via the constant c_0).

6.2 Liouville measure, PD statistics and freezing.

The last statements to be still proved are those dealing with the limit of the Liouville measure, Poisson Dirichlet statistics of the corresponding atomic law and the freezing phenomenon. All of these pertain to $\beta > \beta_c$ where, we recall, $\beta_c = \alpha := 2/\sqrt{g}$. Throughout this section we suppose that $D \in \mathfrak{D}$ and a sequence $\{D_N\}$ satisfying (2.1–2.2) are given and fixed.

Fix $\beta > \beta_c$ and recall the definition of $\Sigma_{s,Q}$ from before Theorem 2.6. Our principal goal is to prove that, for every bounded and continuous function $f: D \rightarrow [0, \infty)$,

$$\sum_{z \in D_N} e^{\beta(h^{D_N}(z) - m_N)} f(x/N) \xrightarrow[N \rightarrow \infty]{\text{law}} c(\beta) Z^D(D)^{\beta/\beta_c} \int_D \Sigma_{\beta_c/\beta, \widehat{Z}^D}(\mathrm{d}x) f(x). \quad (6.43)$$

A natural approach is to write the left-hand side as $\langle \eta_N^D, \tilde{f} \rangle$, where $\tilde{f}(x, h) := e^{\beta h} f(x)$, and apply Corollary 2.2. It does not bother us much that \tilde{f} is unbounded as we can always truncate the maximum from above with high probability. However, the fact that the support of \tilde{f} extends all the way to $-\infty$ in the h variable is much more serious as this could lead to potential blow-ups, which need to be ruled out before the limit $N \rightarrow \infty$ is taken.

Recall the definition of $\Gamma_N^D(t)$ from (5.57) and for $\delta > 0$ small, set

$$D_N^\delta := \{x \in D_N : \text{dist}(x, D_N^c) > \delta N\} \quad \text{and} \quad \Gamma_{N,\delta}^D(t) := \Gamma_N^D(t) \cap D_N^\delta. \quad (6.44)$$

The said blow-ups will be controlled with the help of the following claim:

Proposition 6.8 *For $D \in \mathfrak{D}$ there is a constant $c \in (0, \infty)$ such that for each $\varepsilon > 0$ small enough, all $N \geq 1$ large and all $t \geq 0$,*

$$P(|\Gamma_N^D(t)| \geq 2e^{(\beta_c + \varepsilon)t}) \leq c(1+t)^2 e^{-\varepsilon t}. \quad (6.45)$$

We begin with a lemma:

Lemma 6.9 *For each $D \in \mathfrak{D}$ and each $\delta > 0$ there is a constant $c \in (0, \infty)$ depending only on δ and the diameter of D such that for all $t, s \geq 0$,*

$$\max_{x \in D_N^\delta} P(h^{D_N}(x) \geq m_N - t, h^{D_N} \leq m_N + s) \leq c(1+s+t)^2 e^{\alpha t} \frac{1}{N^2} \quad (6.46)$$

Proof. For notational convenience suppose that $0 \in D_N^\delta$ and instead of the maximum over $x \in D_N$ in the statement, let us set $x = 0$ and take maximum over all shifts of D_N such that $0 \in D_N^\delta$. The fact that $\delta > 0$ implies that there $n, q \geq 1$, with $n - \log_2 N$ and q bounded by δ -dependent constants uniformly in $N \geq 1$ and all shifts of D_N for which $0 \in D_N^\delta$, such that

$$\Delta^n \subseteq D_N \subseteq \Delta^{n+q}. \quad (6.47)$$

Thanks to Lemma 4.23, we can use Lemma 4.21 with D_N instead of Δ^n and from (4.66) thus get

$$P\left(h^{D_N} \leq m_N + s - (m_N + t) \mathfrak{g}^{D_N} \mid h^{D_N}(0) = 0\right) \leq \frac{c_1}{n} (1+s+t)^2 \quad (6.48)$$

for some $c \in (0, \infty)$. Lemma 3.2 now tells us that the probability on the left equals

$$P(h^{D_N} \leq m_N + s \mid h^{D_N}(0) = m_N - t) \quad (6.49)$$

and the probability in the statement is thus bounded by

$$\frac{c_1}{n} \int_0^{s+t} dx \frac{(1+s+t-x)^2}{\sqrt{2\pi \text{Var}(h^{D_N}(0))}} e^{-\frac{1}{2} \frac{(m_N+x-t)^2}{\text{Var}(h^{D_N}(0))}}. \quad (6.50)$$

It remains to carefully estimate the integral on the right-hand side.

Since $\text{Var}(h^{D_N}(0)) - g \log N$ is bounded by a constant uniformly in N and uniformly in the position of D_N subject to $0 \in D_N^\delta$, we have

$$\frac{1}{\sqrt{\text{Var}(h^{D_N}(0))}} e^{-\frac{1}{2} m_N^2 / \text{Var}(h^{D_N}(0))} \leq c_2 \frac{\log N}{N^2} \quad (6.51)$$

for some $c_2 \in (0, \infty)$. Expanding the square in the exponent and using that $m_N / \text{Var}(h^{D_N}(0)) \geq \alpha - c_3(\log \log N) / \log N$ for some $c_3 \in (0, \infty)$ we bound

$$\frac{m_N}{\text{Var}(h^{D_N}(0))} (x-t) + \frac{1}{2} \frac{(x-t)^2}{\text{Var}(h^{D_N}(0))} \geq \alpha(x-t) - c_3(x-t) \frac{\log \log N}{\log N} + c_4 \frac{(x-t)^2}{\log N} \quad (6.52)$$

for some $c_4 \in (0, \infty)$. The last two terms are minimized by $x-t$ of order $\log \log N$ and so the right-hand side is at least $\alpha(x-t) - c_5$ for some $c_5 \in (0, \infty)$. Since n is order $\log N$, (6.50) is bounded by

$$\frac{c_6}{N^2} (1+s+t)^2 \int_0^{s+t} e^{-\alpha(x-t)} dx \quad (6.53)$$

for some $c_6 \in (0, \infty)$. The claim now follows by simple integration. \square

Proof of Proposition 6.8. Let $\delta > 0$ be fixed small and let $\tilde{D} \supset D$ be such that $\tilde{D}_N^\delta \supset D_N$ holds for all $N \geq 1$. From Lemma 6.9 we get

$$E(|\Gamma_{N,\delta}^{\tilde{D}}(t)| 1_{\{h^{D_N} \leq m_N+s\}}) \leq c_1 (1+s+t)^2 e^{\alpha t}. \quad (6.54)$$

Lemma B.10 then allows us to estimate

$$\begin{aligned} P(|\Gamma_N^D(t)| \geq 2e^{(\alpha+\varepsilon)t}) &\leq 2P(|\Gamma_{N,\delta}^{\tilde{D}}(t)| \geq e^{(\alpha+\varepsilon)t}) \\ &\leq 2P(\max h^{\tilde{D}_N} \leq m_N+s, |\Gamma_{N,\delta}^{\tilde{D}}(t)| \geq e^{(\alpha+\varepsilon)t}) + P(\max h^{\tilde{D}_N} > m_N+s) \\ &\leq 2c_1 (1+s+t)^2 e^{-\varepsilon t} + 2c_2 (1+s) e^{-\alpha s}, \end{aligned} \quad (6.55)$$

where we used the Markov inequality in the first term and Lemma B.13 in the second. Setting $s := t$ then yields the desired claim. \square

This will permit us to complete:

Proof of Theorem 2.6. Let $\beta > \beta_c$ and let $\varepsilon \in (0, \beta - \beta_c)$. In light of the argument at the beginning of this section, the key point is to reduce the sum on the left of (6.43) to those z where $|h_z^{D_N} - m_N|$ is bounded uniformly in N . This is done as follows: By Proposition 6.8 and a union bound, there are constants $c, \tilde{c} \in (0, \infty)$ such that

$$|\Gamma_N^D(t)| \leq c e^{(\beta_c + \varepsilon)t}, \quad t \geq t_0, \quad (6.56)$$

occurs with probability at least $1 - \tilde{c}e^{-\varepsilon t_0/2}$ uniformly in $N \geq 1$. When (6.56) is in force, we have

$$\begin{aligned} \sum_{z \in D_N} e^{\beta(h^{D_N}(z) - m_N)} \mathbf{1}_{\{h^{D_N}(z) \leq m_N - t_0\}} \\ \leq \sum_{n \geq 0} e^{-\beta(t_0 + n)} |\Gamma_{N,\delta}^D(t_0 + n + 1)| \leq c' e^{(\beta_c + \varepsilon - \beta)t_0} \end{aligned} \quad (6.57)$$

which is small for t_0 large by our choice of ε . On the other hand, by Lemma B.13, the probability that there is any $z \in D_N$ where $h^{D_N}(z) - m_N \geq t$ is small in t , uniformly in N . It follows that

$$\lim_{t_0 \rightarrow \infty} \limsup_{N \rightarrow \infty} P\left(\sum_{z \in D_N} e^{\beta(h^{D_N}(z) - m_N)} f(x/N) \mathbf{1}_{\{h^{D_N}(z) - m_N \notin [-t_0, t_0]\}} > \delta'\right) = 0 \quad (6.58)$$

holds for each $\delta' > 0$.

We will now invoke the full process convergence to control the contribution of the points where $|h^{D_N}(x) - m_N| \leq t_0$. Instead of Corollary 2.2, it will be more convenient to aim directly at Theorem 2.1. First we invoke Lemma B.11 which says that the probability that $|h^{D_N}(z) - m_N| \leq t_0$ at $z = x, y$ with $r|x - y| \leq N/r$ tends to zero as $N \rightarrow \infty$ and $r \rightarrow \infty$. Thanks to uniform continuity of f , defining $Y_r^\beta(\phi) := \sum_{z \in \Lambda_r(0)} e^{-\beta\phi(x)}$ this implies

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} \limsup_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P\left(\left|\sum_{z \in D_N} e^{\beta(h^{D_N}(z) - m_N)} f(x/N) \right. \right. \\ \left. \left. - \sum_{x \in \Theta_{N,r}} e^{\beta(h^{D_N}(x) - m_N)} Y_r(h_x^{D_N} - h_{x+}^{D_N}) f(x/N) \mathbf{1}_{\{h^{D_N}(x) - m_N \in [-t_0, t_0]\}} \right| > \delta\right) = 0, \end{aligned} \quad (6.59)$$

where we also invoked (6.58) in order to be able to write only an indicator involving $h_x^{D_N}$ in the second sum. This sum can now be written as $\langle \eta_{N,r}^D, F_{r,t_0} \rangle$, where

$$F_{r,t_0}(x, h, \phi) := f(x) e^{\beta h} \mathbf{1}_{\{|h| \leq t_0\}} Y_r^\beta(\phi). \quad (6.60)$$

Thanks to (6.59), the limit proved in Theorem 2.1 now shows

$$\begin{aligned} E\left(\exp\left\{-\sum_{z \in D_N} e^{\beta(h^{D_N}(z) - m_N)} f(x/N)\right\}\right) \\ \xrightarrow[N \rightarrow \infty]{r \rightarrow \infty} \lim_{r \rightarrow \infty} E\left(\exp\left\{-\int Z^D(dx) \otimes e^{-\alpha h} dh \otimes \nu(d\phi) (1 - e^{-F_r(x, h, \phi)})\right\}\right), \end{aligned} \quad (6.61)$$

where, with the help of the Monotone Convergence Theorem, we already took the limit $t_0 \rightarrow \infty$ inside the expectation and replaced F_{r,t_0} by

$$F_r(x, h, \phi) := f(x) e^{\beta h} Y_r^\beta(\phi). \quad (6.62)$$

Note that the integral on the right of (6.61) is finite a.s. because $Z^D(D) < \infty$ a.s. and $1 - e^{-F_r(x, h, \phi)}$ decays proportionally to $e^{\beta h}$ as $h \rightarrow -\infty$ and $\beta > \alpha$.

We now wish to address the limit $r \rightarrow \infty$ in (6.61). Focusing first on the integral on the right-hand side, the change of variables $t := e^{\beta h}$ gives us

$$\begin{aligned} \int e^{-\alpha h} dh (1 - e^{-F_r(x, h, \phi)}) &= \int_0^\infty \frac{dt}{\beta t} t^{-\alpha/\beta} (1 - e^{-f(x) Y_r^\beta(\phi) t}) \\ &= \frac{1}{\beta} (Y_r^\beta(\phi))^{\alpha/\beta} \int_0^\infty dt t^{-1-\alpha/\beta} (1 - e^{-t f(x)}). \end{aligned} \quad (6.63)$$

By our arguments above, the left-hand side of (6.61) is positive for $f(x) := 1$, and since $Z^D(D) > 0$ a.s., the right-hand side of (6.63) must remain bounded as $r \rightarrow \infty$ even after taking expectation with respect to ν . The Monotone Convergence Theorem then gives $E_\nu(Y^\beta(\phi)^{\alpha/\beta}) < \infty$ and, in particular, $Y(\phi) < \infty$ ν -a.s. The $r \rightarrow \infty$ limit now can be taken inside (6.61) thus replacing F_r by

$$F(x, h, \phi) = f(x) e^{\beta h} Y^\beta(\phi). \quad (6.64)$$

It remains to identify the resulting expression on the right-hand side of (6.61) with the Laplace transform of the right-hand side of (6.43).

The definition of $\Sigma_{s,Q}$ reads

$$E\left(\exp\left\{-\lambda \int \Sigma_{s,Q}(\mathrm{d}x) f(x)\right\}\right) = \exp\left\{-\lambda^s \int Q(\mathrm{d}x) \otimes \mathrm{d}t t^{-1-s} (1 - e^{-tf(x)})\right\} \quad (6.65)$$

Setting

$$s := \alpha/\beta, \quad \lambda^s := \beta^{-1} E_\nu(Y^\beta(\phi)^{\alpha/\beta}) Z^D(D) \quad \text{and} \quad Q := \widehat{Z}^D \quad (6.66)$$

(6.63) identifies the exponential on the right-hand side of (6.61) (with F_r replaced by F) with that on the right of (6.65). Taking expectation with respect to Z^D yields (6.43) with $c(\beta)$ as given in (2.21). \square

Proof of Corollary 2.7. By using test functions of the form $\lambda_1 + \lambda_2 f(x)$ in the convergence in Theorem 2.6 we find out that

$$\left(\sum_{z \in D_N} e^{\beta(h_z^{D_N} - m_N)}, \sum_{z \in D_N} e^{\beta(h_z^{D_N} - m_N)} f(x/N) \right) \xrightarrow[N \rightarrow \infty]{\text{law}} c(\beta) Z^D(D)^{\beta/\beta_c} \left(\Sigma_{\beta_c/\beta, \widehat{Z}^D}(D), \int_D \Sigma_{\beta_c/\beta, \widehat{Z}^D}(\mathrm{d}x) f(x) \right). \quad (6.67)$$

The claim follows by dividing these two terms and noting that if $\{q_i\}$ are the points of the Poisson process defining $\Sigma_{s,Q}$ as in (2.18), then $\Sigma_{s,Q}(D) = \sum_i q_i$ and $\{q_i / \sum_j q_j : i \geq 1\}$, reordered according to size, constitute a sample from $\text{PD}(s)$. \square

Proof of Corollary 2.8. Recall the function $G_{N,\beta}(t)$ from (2.26). Using the same arguments as in the proof of Theorem 2.6, setting $f(h, \phi) := e^{\beta(h-y)} Y^\beta(\phi)$ yields

$$G_{N,\beta}(y + m_N) \xrightarrow[N \rightarrow \infty]{} E\left(\exp\left\{-Z^D(D) \int e^{-\alpha h} \mathrm{d}h \otimes \nu(\mathrm{d}\phi) (1 - e^{-f(h,\phi)})\right\}\right). \quad (6.68)$$

By change of variables again,

$$\begin{aligned} \int e^{-\alpha h} \mathrm{d}h (1 - e^{-f(h,\phi)}) &= \int_0^\infty \frac{\mathrm{d}t}{\beta t} t^{-\alpha/\beta} (1 - e^{-e^{-\beta y} Y^\beta(\phi) t}) \\ &= \frac{1}{\beta} (e^{-\beta y} Y^\beta(\phi))^{\alpha/\beta} \int_0^\infty \mathrm{d}t t^{-1-\alpha/\beta} (1 - e^{-t}). \end{aligned} \quad (6.69)$$

Defining

$$\tilde{c}(\beta) := \frac{1}{\alpha} \log\left(\frac{1}{\beta} E_\nu(Y(\theta)^{\alpha/\beta}) \int_0^\infty \mathrm{d}t t^{-1-\alpha/\beta} (1 - e^{-t})\right), \quad (6.70)$$

we then get

$$G_{N,\beta}(y + m_N + \tilde{c}(\beta)) \xrightarrow[N \rightarrow \infty]{} E(e^{-Z^D(D) e^{-\alpha y}}) \quad (6.71)$$

exactly as desired. \square

APPENDIX A: BROWNIAN PATHS ABOVE A CURVE

The goal of this section is prove Propositions 4.7, 4.8, 4.9, 4.10, 4.13 and 4.14 dealing with probabilities that Brownian motion and Brownian bridge avoid hitting a given curve. Many of the calculations presented here appear in some form in other places including the literature on the DGFF and the Branching Brownian Motion. However, as discussed in Remark 4.11, our applications require a level of precision and generality that forces us to furnish independent proofs.

A.1 Consequences of the Reflection Principle.

Let $\{B_t : t \geq 0\}$ be a standard Brownian motion. For a continuous function $g : [0, \infty) \rightarrow [0, \infty)$, define

$$\tau_g := \inf\{s \geq 0 : B_s - g(s) = 0\}. \quad (\text{A.1})$$

(In particular, for $g(s) := x$ we will write τ_x and for $g(s) := x + \zeta(s)$ we will write $\tau_{x+\zeta}$, etc.) We begin by dealing with Brownian motion and Brownian bridge above a *constant* curve which boils down to standard applications of the Reflection Principle. The following specific facts will be quite handy in various calculations below:

Lemma A.1 *For all $x > 0$ and all $t > 0$,*

$$P^0(\tau_x \in dt) \leq \frac{1}{\sqrt{2\pi}} \frac{x}{t^{3/2}} dt \quad (\text{A.2})$$

and

$$\sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}} \left(1 - \frac{x^2}{2t}\right) \leq P^0(\tau_x > t) \leq \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}}. \quad (\text{A.3})$$

Similarly, for each $t > 0$ and each $x, y > 0$,

$$\left(1 - \frac{xy}{t}\right) \frac{2xy}{t} \leq P^x(\tau_0 > t \mid B_t = y) \leq \frac{2xy}{t}. \quad (\text{A.4})$$

Finally, let $M_t^* := \max_{s \leq t} B_s$ and $T_t^* := \sup\{s \leq t : B_s = M_t^*\}$. Then $B_{T_t^*} = M_t^*$ and

$$P^0(T_t^* \in ds, M_t^* \in dz) = \frac{ze^{-\frac{z^2}{2s}}}{\pi s^{3/2} \sqrt{t-s}} 1_{\{0 \leq s \leq t\}} 1_{\{z \geq 0\}} ds dz. \quad (\text{A.5})$$

Proof. These claims are standard and can be found (albeit perhaps not in such compact form) in various textbooks. We provide proofs for completeness of exposition.

The Reflection Principle shows

$$P^0(\tau_x > t) = P^0(|B_1| \leq \frac{x}{\sqrt{t}}). \quad (\text{A.6})$$

Then (A.2–A.3) follow by bounding the probability density of B_1 on $[-\frac{x}{\sqrt{t}}, \frac{x}{\sqrt{t}}]$ by $(2\pi)^{-1/2}$ from above and by $(2\pi)^{-1/2}(1 - \frac{x^2}{2t})$ from below. Another application of the Reflection Principle yields

$$P^x(\tau_0 > t \mid B_t = y) = 1 - \exp\left\{-\frac{2yx}{t}\right\}. \quad (\text{A.7})$$

The bounds (A.4) again follow from $a - \frac{a^2}{2} \leq 1 - e^{-a} \leq a$ valid for all $a \geq 0$. For (A.5) we note that, by the Strong Markov Property and path-continuity of the Brownian motion,

$$P^0(T_t^* > s, M_t^* \geq z) = \lim_{\varepsilon \downarrow 0} \sum_{k \geq 0} \int_{(s, t]} P^0(\tau_{z+k\varepsilon} \in du) P^0(\tau_\varepsilon > t - u). \quad (\text{A.8})$$

Using that (A.3) is sharp in the limit $x \downarrow 0$, this shows

$$P^0(T_t^* > s, M_t^* \geq z) = \sqrt{\frac{2}{\pi}} \int_{[z, \infty)} d\tilde{z} \int_{(s, t]} P^0(\tau_{\tilde{z}} \in du) \frac{1}{\sqrt{t-u}}. \quad (\text{A.9})$$

The result then follows by (A.6) and differentiation. That $B_{T_t^*} = M_t^*$ holds is a consequence of path continuity of the Brownian motion. \square

A.2 Positive curves: Brownian motion.

Our next task is the control of Brownian motion above a positive curve. The following is key in the proof of Proposition 4.7:

Proposition A.2 *Let $\zeta : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing, continuous and obeys $\zeta(s) = o(s^{1/2})$ as $s \rightarrow \infty$. Then for all $x > \zeta(0)$ and all $t > 0$,*

$$P^0(\tau_{x-\zeta} \leq t < \tau_x) \leq 2 \frac{\rho(x)^{2/3} x^{1/3}}{\sqrt{t}}, \quad (\text{A.10})$$

where $\rho(x)$ be the quantity from (4.23).

This is indeed the case, as we see from:

Proof of Proposition 4.7. We have

$$P^0(\tau_x > t) = P^0(\tau_{x-\zeta} > t) + P^0(\tau_{x-\zeta} \leq t < \tau_x) \quad (\text{A.11})$$

To get the claim we just subtract (A.10) from the left-hand side of (A.3). \square

Also the proof of Proposition A.2 begins by a slightly stricter estimate:

Lemma A.3 *Let ζ and ρ be as above. Then for all $x > \zeta(0)$ and all $\delta > 0$,*

$$P^0(\tau_{x-\zeta} < t, \tau_x > (1+\delta)t) \leq \frac{1}{\sqrt{\delta}} \frac{\rho(x)}{\sqrt{t}}. \quad (\text{A.12})$$

Proof. A routine approximation argument (based on path continuity of the Brownian motion) permits us to assume that ζ is continuously differentiable. Using the Strong Markov Property around the stopping time $\tau_{x-\zeta}$ yields

$$P^0(\tau_{x-\zeta} < t, \tau_x > (1+\delta)t) = \int_0^t P^0(\tau_{x-\zeta} \in ds) P^0(\tau_{\zeta(s)} > (1+\delta)t - s). \quad (\text{A.13})$$

By (A.3), the second probability is at most $\zeta(s)/\sqrt{\delta t}$ for all $s \in [0, t]$. Integrating by parts and using positivity of $\zeta(t)$, the integral is then at most

$$\frac{1}{\sqrt{t}} \frac{1}{\sqrt{\delta}} \left[\zeta(0) + \int_0^t \zeta'(s) P^0(\tau_{x-\zeta} > s) ds \right]. \quad (\text{A.14})$$

Our goal is to show that the term in the brackets is bounded by $\rho(x)$. For this we first take $t \rightarrow \infty$ and then bound $P^0(\tau_{x-\zeta} > s)$ by $P^0(\tau_x > s)$. Then we integrate by parts and use that $\zeta(s) = o(s^{1/2})$ (because otherwise $\rho(x) = \infty$) to find that the bracket is bounded by the left-hand side of

$$\int_0^\infty P^0(\tau_x \in ds) \zeta(s) \leq \zeta(x^2) + \frac{x}{2} \int_{x^2}^\infty \frac{\zeta(s)}{s^{3/2}} ds. \quad (\text{A.15})$$

Here, to get the bound on the right, we split the integral at $s := x^2$, invoked the monotonicity of ζ in the first one while applied (A.3) and also used $\sqrt{2\pi} \geq 2$ in the second one. \square

Proof of Proposition A.2. We have

$$P^0(\tau_{x-\zeta} < t < \tau_x) \leq P^0(\tau_{x-\zeta} < t, \tau_x > (1+\delta)t) + P^0(t \leq \tau_x < t \leq (1+\delta)t). \quad (\text{A.16})$$

Using (A.2) and (A.12), we can bound the right-hand side by

$$\frac{1}{\sqrt{\delta}} \frac{\rho(x)}{\sqrt{t}} + \frac{1}{\sqrt{2\pi}} \frac{x\delta}{\sqrt{t}}. \quad (\text{A.17})$$

Bounding $\sqrt{2\pi} \geq 2$, the resulting expression is, as a function of δ , minimized at $\delta := (\rho(x)/x)^{2/3}$. Plugging this back in, we get the claim. \square

A.3 Positive curves: Brownian bridge.

We will now move to the case of the Brownian bridge above a positive curve. The goal is to prove Proposition 4.8 whose key part is the following claim:

Proposition A.4 *Let $\zeta: [0, \infty) \rightarrow [0, \infty)$ be non-decreasing, continuous and obeys $\zeta(s) = o(s^{1/2})$ as $s \rightarrow \infty$. Then for all $x, y > \zeta(0)$ and all $t > 0$,*

$$\begin{aligned} P^x\left(\min_{0 \leq s \leq t} [B_s - \zeta(s \wedge (t-s))] < 0 < \min_{0 \leq s \leq t} B_s \mid B_t = y\right) \\ \leq 8 \left(\sqrt{\frac{\rho(x)}{x}} + \sqrt{\frac{\rho(y)}{y}} \right) \frac{xy}{t} e^{\frac{(x-y)^2}{2t}}, \end{aligned} \quad (\text{A.18})$$

where $\rho(x)$ is the quantity from (4.23).

We first directly check that this indeed implies the desired bound:

Proof of Proposition 4.8. We just combine (A.18) with the bound on the left of (A.4). \square

The proof of Proposition A.4 will be based on Lemma A.3 and some interesting technical ingredients. The first one is inspired by arguments from the proof of Bramson [14, Proposition 1']:

Lemma A.5 (Decoupling lemma) *Pick $x, y \in \mathbb{R}$, let $t, t_1, t_2 > 0$ be such that $t_1 + t_2 < t$ and let A_1 and A_2 be events such that $A_i \in \sigma(B_s: 0 \leq s \leq t_i)$, $i = 1, 2$. Consider also the event*

$$A'_2 := \{ \text{path } \{B_{t-s}: 0 \leq s \leq t_2\} \text{ lies in } A_2 \}. \quad (\text{A.19})$$

Then

$$P^x(A_1 \cap A'_2 \mid B_t = y) \leq \sqrt{\frac{t}{t-t_1-t_2}} e^{\frac{(x-y)^2}{2t}} P^x(A_1) P^y(A_2). \quad (\text{A.20})$$

Proof. Define the functions

$$f_1(z) := P^x(A_1 \mid B_{t_1} = z) \quad \text{and} \quad f_2(z) := P^y(A_2 \mid B_{t_2} = z). \quad (\text{A.21})$$

Conditioning on B_{t_1} and B_{t-t_2} and invoking the reversibility of the Brownian bridge then yields

$$P^x(A_1 \cap A'_2 \mid B_t = y) = E^x(f_1(B_{t_1}) f_2(B_{t-t_2}) \mid B_t = y). \quad (\text{A.22})$$

Now let $p_{t_1, t-t_2}^{x,y}(x_1, x_2)$ denote the joint probability density of (B_{t_1}, B_{t-t_2}) in measure $P^x(-|B_t = y)$ and let $g_t(x)$ denote the probability density of a normal random variable with mean zero and variance t . Then

$$p_{t_1, t-t_2}^{x,y}(x_1, x_2) = \frac{g_{t_1}(x_1 - x) g_{t-t_1-t_2}(x_2 - x_1) g_{t_2}(y - x_2)}{g_t(y - x)}. \quad (\text{A.23})$$

Using the explicit form of g_t in the denominator and the bound $g_t(x) \leq \frac{1}{\sqrt{2\pi t}}$ for the middle term in the numerator gives

$$p_{t_1, t-t_2}^{x,y}(x_1, x_2) \leq \sqrt{\frac{t}{t-t_1-t_2}} e^{\frac{(x-y)^2}{2t}} g_{t_1}(x_1 - x) g_{t_2}(y - x_2). \quad (\text{A.24})$$

Since both f_1 and f_2 are positive, plugging this in (A.22) then yields the claim. \square

Lemma A.5 permits us to effectively “tear” a Brownian bridge apart into two independent Brownian paths. The next lemma will in turn let us symmetrize the events about the midpoint of the interval, and also assume that the starting and ending points of the Brownian bridge are the same. This is useful in absorbing a “vertical shift” in the change of the endpoint values.

Lemma A.6 (Symmetrization lemma) *Let $t > 0$ be given and let $A_1, A_2 \in \sigma(B_s : 0 \leq s \leq t/2)$. Let A'_1 , resp., A'_2 be related to A_1 , resp., A_2 as in (A.19). Then for all $x, y \in \mathbb{R}$,*

$$P^x(A_1 \cap A'_2 | B_t = y) \leq P^x(A_1 \cap A'_1 | B_t = x)^{1/2} P^y(A_2 \cap A'_2 | B_t = y)^{1/2} e^{\frac{(x-y)^2}{2t}}. \quad (\text{A.25})$$

Proof. Denote

$$f_{x,y}(z) := \sqrt{\frac{2}{\pi t}} e^{-\frac{2}{t}(z - \frac{x+y}{2})^2}. \quad (\text{A.26})$$

As is easy to check, $f_{x,y}$ is the (probability) density of $P^x(B_{t/2} \in \cdot | B_t = y)$ with respect to the Lebesgue measure. Therefore,

$$P^x(A_1 \cap A'_2 | B_t = y) = \int_{\mathbb{R}} dz f_{x,y}(z) P^x(A_1 | B_{t/2} = z) P^y(A_2 | B_{t/2} = z). \quad (\text{A.27})$$

A calculation now shows

$$f_{x,y}(z) = f_{x,x}(z)^{1/2} f_{y,y}(z)^{1/2} e^{\frac{(x-y)^2}{2t}}. \quad (\text{A.28})$$

The claim follows by plugging this in the above integral, invoking the Cauchy-Schwarz inequality and wrapping the result together using again (A.27). \square

Proof of Proposition A.4. Consider the events

$$A_1 := \left\{ \min_{s \leq t/2} (B_s - \zeta(s)) < 0 < \min_{s \leq t/2} B_s \right\} \quad \text{and} \quad A_2 := \left\{ \min_{s \leq t/2} B_s > 0 \right\} \quad (\text{A.29})$$

Using A' to denote the path reversal of event A as in Lemma A.5, the event in the statement is contained in $(A_1 \cap A'_2) \cup (A_2 \cap A'_1)$. By Lemma A.6, the desired probability is thus at most

$$\begin{aligned} & P^x(A_1 \cap A'_1 | B_t = x)^{1/2} P^y(A_2 \cap A'_2 | B_t = y)^{1/2} \\ & + P^x(A_2 \cap A'_2 | B_t = x)^{1/2} P^y(A_1 \cap A'_1 | B_t = y)^{1/2} \end{aligned} \quad (\text{A.30})$$

times $e^{\frac{(x-y)^2}{2t}}$. By symmetry, it suffices to bound the first term in (A.30) by $8 \sqrt{\frac{\rho(x)}{x} \frac{xy}{t}}$.

We introduce two additional events

$$A_3 := \left\{ \min_{s \leq t/2} (B_s - \zeta(s)) < 0 < \min_{s \leq \frac{3}{4}t} B_s \right\} \quad \text{and} \quad A_4 := \left\{ \min_{s \leq \frac{1}{8}t} B_s > 0 \right\}. \quad (\text{A.31})$$

Then $A_1 \cap A'_1 \subseteq A_3 \cap A'_4$ and so, by Lemma A.5 with $t_1 := \frac{3}{4}t$ and $t_2 := \frac{1}{8}t$,

$$P^x(A_1 \cap A'_1 | B_t = x) \leq P^x(A_3 \cap A'_4 | B_t = x) \leq \sqrt{8} P^x(A_3) P^x(A_4). \quad (\text{A.32})$$

We now note that, by Lemma A.3,

$$P^x(A_3) = P^0(\tau_{x-\zeta} < \frac{1}{2}t, \tau_x > \frac{3}{4}t) \leq 2 \frac{\rho(x)}{\sqrt{t}}, \quad (\text{A.33})$$

while Lemma A.1 gives

$$P^x(A_4) = P^0(\tau_x > \frac{1}{8}t) \leq \sqrt{8} \frac{x}{\sqrt{t}} \quad (\text{A.34})$$

and

$$P^y(A_2 \cap A'_2 | B_t = y) = P^y(\tau_0 > t | B_t = y) \leq \frac{2y^2}{t}. \quad (\text{A.35})$$

Combining these facts, we get the desired statement. \square

A.4 Negative curves.

The control of Brownian motion and Brownian bridge above negative curves involves considerably heavier calculations but we will also be able to enjoy the benefits of our earlier work. The following is the main ingredient for the proof of Proposition 4.9:

Proposition A.7 *Let $\zeta: [0, \infty) \rightarrow [0, \infty)$ be non-decreasing, continuous and obeys $\zeta(s) = o(s^{1/4})$ as $s \rightarrow \infty$. Then for all $x > 0$ and all $t > 0$,*

$$P^0(\tau_x < t < \tau_{x+\zeta}) \leq 2 \left(1 + \left(\frac{\tilde{\rho}(x)}{x} \right)^{2/3} \right) \frac{\tilde{\rho}(x)^{2/3} x^{1/3}}{\sqrt{t}}, \quad (\text{A.36})$$

where $\tilde{\rho}(x)$ be the quantity from (4.28).

Proof of Proposition 4.9. Just add (A.36) to the right-hand side of (A.3). \square

The proof of Proposition A.7 will follow the same strategy as for the case of positive curves. We start with slightly stricter version of the desired estimate:

Lemma A.8 *For ζ and $\tilde{\rho}$ as above, any $x > 0$, any $t > 0$ and any $\delta > 0$,*

$$P^0(\tau_x < t, \tau_{x+\zeta} > (1+\delta)t) \leq \frac{1}{\sqrt{\delta}} \frac{\tilde{\rho}(x)}{\sqrt{t}}. \quad (\text{A.37})$$

The proof is an augmented version of the argument from the proof of Lemma A.3. However, we will need to replace the bounds in Lemma A.1 by the following estimate:

Lemma A.9 *For $\zeta: [0, \infty) \rightarrow [0, \infty)$ non-decreasing and continuously differentiable and $t > 0$,*

$$P^0(\tau_\zeta > t) \leq \frac{1}{\sqrt{t}} \left(\zeta(0) + 2 \int_0^t du \frac{\zeta(u) \zeta'(u)}{\sqrt{u}} \right). \quad (\text{A.38})$$

Proof. The proof is inspired by an argument from Bramson's proof of his Proposition 1 in [14]. Recall our earlier definitions $M_t^* := \max_{s \leq t} B_s$ and $T_t^* := \sup\{s \leq t : B_s = M_t^*\}$. By (A.5) and (A.3), we have

$$\begin{aligned} P^0(\tau_\zeta > t) &\leq P^0(\tau_{\zeta(0)} > t) + P^0(\zeta(T_t^*) > M_t^* \geq \zeta(0)) \\ &\leq \frac{\zeta(0)}{\sqrt{t}} + \int_0^t ds \int_{\zeta(0)}^{\zeta(s)} dz \frac{ze^{-\frac{z^2}{2s}}}{\pi s^{3/2} \sqrt{t-s}}. \end{aligned} \quad (\text{A.39})$$

We need to estimate the integral on the right. By way of a routine approximation argument we may assume that ζ is invertible and ζ^{-1} thus exists on $[\zeta(0), \zeta(t)]$. This permits us to write

$$\int_0^t ds \int_{\zeta(0)}^{\zeta(s)} dz \frac{ze^{-\frac{z^2}{2s}}}{\pi s^{3/2} \sqrt{t-s}} = \int_{\zeta(0)}^{\zeta(t)} dz \int_{\zeta^{-1}(z)}^t ds \frac{ze^{-\frac{z^2}{2s}}}{\pi s^{3/2} \sqrt{t-s}}. \quad (\text{A.40})$$

We now estimate the inner integral by splitting the domain around the point $(t/2) \wedge \zeta^{-1}(z)$. After some straightforward calculations, the double integral in (A.40) is bounded by

$$\frac{4\sqrt{2}}{\pi\sqrt{t}} \int_{\zeta(0)}^{\zeta(t)} dz \frac{z}{\sqrt{\zeta^{-1}(z)}}. \quad (\text{A.41})$$

Substituting $z := \zeta(s)$ and using that $4\sqrt{2} \leq 2\pi$ we then readily get the claim. \square

Proof of Lemma A.8. As before, approximation arguments permit us to assume that ζ is continuously differentiable. As in the proof of Lemma A.3, we write

$$P^0(\tau_x < t, \tau_{x+\zeta} > (1+\delta)t) = \int_{[0,t]} P^0(\tau_x \in ds) P^0(\tau_{\zeta(s+)} > (1+\delta)t - s). \quad (\text{A.42})$$

Using that $(1+\delta)t - s \geq \delta t$ throughout the domain of integration, we then use Lemma A.9 to bound the (last) probability on the right by $f(s)/\sqrt{\delta t}$, where

$$f(s) := \zeta(s) + 2 \int_0^\infty du \frac{\zeta(s+u)\zeta'(s+u)}{\sqrt{u}}. \quad (\text{A.43})$$

The contribution of the first term on the right-hand side is handled by the argument in the proof of Lemma A.3 (specifically, (A.15)). Using also (A.2), we thus get

$$\begin{aligned} P^0(\tau_x < t, \tau_{x+\zeta} > (1+\delta)t) &\leq \frac{\rho(x)}{\sqrt{\delta t}} \\ &\quad + \frac{2}{\sqrt{2\pi\delta t}} \int_0^\infty ds \int_0^\infty du \frac{x}{s^{3/2}} e^{-\frac{x^2}{2s}} \frac{\zeta(s+u)\zeta'(s+u)}{\sqrt{u}}. \end{aligned} \quad (\text{A.44})$$

We now perform the substitution $w := s+u$ and $v := u/s$. The Jacobian of the transformation equals $w/(1+v)^2$. After some cancelations, the double integral in (A.44) thus becomes

$$\int_0^\infty dw e^{-\frac{x^2}{2w}} \frac{x\zeta(w)\zeta'(w)}{w} \int_0^\infty dv \frac{1}{\sqrt{v}} e^{-\frac{x^2}{2w}v}. \quad (\text{A.45})$$

The inner integral evaluates to $\sqrt{2\pi w}/x$ and so the last term on the right-hand side of (A.44) is bounded by $1/\sqrt{\delta t}$ times

$$2 \int_0^\infty ds e^{-\frac{x^2}{2s}} \frac{\zeta(s)\zeta'(s)}{\sqrt{s}} = \frac{1}{2} \int_0^\infty ds \zeta(s)^2 e^{-\frac{x^2}{2s}} \left(\frac{x^2}{s^{5/2}} + s^{-3/2} \right), \quad (\text{A.46})$$

where we integrated by parts and used that $\zeta(s) = o(s^{1/4})$ as $s \rightarrow \infty$. We thus have to show that (A.46) is bounded by the sum of the last two terms in the definition of $\tilde{\rho}$ in (4.28).

Consider the integral on the right of (A.46). We first absorb $x^2/(2s)$ from the first term in the parenthesis at the cost of changing “2” to “4” in the denominator of the exponent of the exponential. The right-hand side of (A.46) is then bounded by

$$\frac{1}{2}(4e^{-1} + 1) \int_0^\infty ds \frac{\zeta(s)^2}{s^{3/2}} e^{-\frac{x^2}{4s}}. \quad (\text{A.47})$$

We split the integral at $s := x^2$ and, in the part corresponding to $s \in [0, x^2]$, use $s^{-1} e^{-\frac{x^2}{4s}} \leq 4e^{-1} x^{-2}$ to bound the expression in (A.47) by

$$2e^{-1}(4e^{-1} + 1) \frac{1}{x^2} \int_0^{x^2} ds \frac{\zeta(s)^2}{s^{1/2}} + \frac{1}{2}(4e^{-1} + 1) \int_{x^2}^\infty ds \frac{\zeta(s)^2}{s^{3/2}}. \quad (\text{A.48})$$

Using $\zeta(s) \leq \zeta(x^2)$ inside the first integral, the result follows by elementary calculations. \square

Proof of Proposition A.7. Fix $\delta := (\tilde{\rho}(x)/x)^{2/3}$ and note that

$$P^0\left(\tau_x < (1 + \delta)t < \tau_{x+\zeta}\right) \leq P^0\left(\tau_x < t, \tau_{x+\zeta} > (1 + \delta)t\right) + P^0\left(t < \tau_x \leq (1 + \delta)t\right). \quad (\text{A.49})$$

As in the proof of Proposition A.2, the right-hand side is bounded using (A.2) and (A.37) by an expression that evaluates to $2\tilde{\rho}(x)^{2/3}x^{1/3}/\sqrt{t}$. The claim follows by relabeling $(1 + \delta)t$ for t . \square

Moving over to the case of Brownian bridge, the key estimate in the proof of Proposition 4.10 is as follows:

Proposition A.10 *Let $\zeta : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing, continuous and obeys $\zeta(s) = o(s^{1/4})$ as $s \rightarrow \infty$. Then for all $x, y > 0$ and all $t > 0$,*

$$\begin{aligned} P^x\left(\min_{0 \leq s \leq t} B_s < 0 < \min_{0 \leq s \leq t} [B_s + \zeta(s \wedge (t-s))]\right) \Big| B_t = y \\ \leq 96 \left(1 + \frac{\tilde{\rho}(x)}{x}\right) \left(1 + \frac{\tilde{\rho}(y)}{y}\right) \left(\sqrt{\frac{\tilde{\rho}(x)}{x + \tilde{\rho}(x)}} + \sqrt{\frac{\tilde{\rho}(y)}{y + \tilde{\rho}(y)}}\right) \frac{xy}{t} e^{-\frac{(x-y)^2}{2t}} \end{aligned} \quad (\text{A.50})$$

where $\tilde{\rho}$ is the quantity from (4.28).

Proof. The proof follows closely that of Proposition A.4. The events A_1 and A_2 are now given by

$$A_1 := \left\{ \min_{s \leq t/2} B_s < 0 < \min_{s \leq t/2} [B_s + \zeta(s)] \right\} \quad \text{and} \quad A_2 := \left\{ \min_{s \leq t/2} [B_s + \zeta(s)] > 0 \right\} \quad (\text{A.51})$$

and the symmetrization argument from Lemma A.6 again reduces the problem to bounding the probabilities of $A_1 \cap A'_1$ and $A_2 \cap A'_2$. Denoting

$$A_3 := \left\{ \min_{s \leq t/2} B_s < 0 < \min_{s \leq \frac{3}{4}t} [B_s + \zeta(s)] \right\} \quad \text{and} \quad A_4 := \left\{ \min_{s \leq \frac{1}{8}t} [B_s + \zeta(s)] > 0 \right\} \quad (\text{A.52})$$

from the monotonicity of ζ we have $A_1 \cap A'_1 \subseteq A_3 \cap A'_4$ while $A_2 \cap A'_2 \subseteq A_4 \cap A'_4$. The decoupling argument in Lemma A.5 then yields

$$P^x(A_1 \cap A'_1 | B_t = x) \leq \sqrt{8} P^x(A_3) P^x(A_4) \quad \text{and} \quad P^y(A_2 \cap A'_2 | B_t = y) \leq 2 P^y(A_4)^2. \quad (\text{A.53})$$

Lemma A.8 now shows

$$P^x(A_3) = P^0(\tau_x < t/2, \tau_{x+\zeta} > \frac{3}{4}t) \leq 2 \frac{\tilde{\rho}(x)}{\sqrt{t}} \quad (\text{A.54})$$

while for $P^y(A_4)$ we get

$$\begin{aligned} P^y(A_4) &= P^0(\tau_{y+\zeta} > \frac{1}{8}t) = P^0(\tau_y > \frac{1}{12}t) + P^0(\tau_y < \frac{1}{12}t, \tau_{y+\zeta} > \frac{1}{8}t) \\ &\leq \sqrt{12} \frac{y}{\sqrt{t}} + \sqrt{2} \sqrt{12} \frac{\tilde{\rho}(y)}{\sqrt{t}} \leq 12 \left(1 + \frac{\tilde{\rho}(y)}{y}\right) \frac{y}{\sqrt{t}}. \end{aligned} \quad (\text{A.55})$$

To get the desired conclusion, just plug these in (A.53) and use (A.30) along with some straightforward algebraic manipulations. \square

Finally, we use the above to dismiss our earlier claim:

Proof of Proposition 4.10. Just combine the right-hand sides of the bounds (A.4) and (A.50). \square

A.5 Entropic repulsion.

The above results permit us to give the proofs of Propositions 4.13 and 4.14 dealing with the phenomenon of entropic repulsion. We begin by proving the statement in Propositions 4.13 for unconditioned Brownian motion. First we show that, on the said event, the Brownian path is already quite high at time u :

Lemma A.11 *For ζ as in Proposition 4.13 there is a constant $c_1 = c_1(a, \sigma) > 0$ such that for all sufficiently large $t > 0$, all $u \in [0, t/4]$ and all $x \geq \zeta(u) \vee 1$,*

$$P^0\left(\min_{0 \leq s \leq t} [B_s + \zeta(s)] > 0, B_u \leq x\right) \leq c_1 \frac{x^2}{\sqrt{u}} \frac{1}{\sqrt{t}}. \quad (\text{A.56})$$

Proof. For any $u \in [0, t]$, abbreviate

$$A_u := \left\{ \min_{u \leq s \leq t} [B_s + \zeta(s)] > 0 \right\}. \quad (\text{A.57})$$

The event in the statement can then be written as $A_0 \cap \{B_u \leq x\}$. From $A_0 \subseteq A_u$ we have

$$P^0(A_0 \cap \{B_u \leq x\}) \leq E\left(1_{\{-\zeta(u) < B_u \leq x\}} P^0(A_u \mid \sigma(B_u))\right). \quad (\text{A.58})$$

Introducing $\zeta_u(s) := \zeta(u+s)$, on the event $\{B_u = x_1\}$ we then have

$$P^0(A_u \mid \sigma(B_u)) = P^{x_1}\left(\min_{0 \leq s \leq t-u} [B_u + \zeta_u(s)] > 0\right). \quad (\text{A.59})$$

A straightforward monotonicity argument then shows that, for $x_1 \in (-\zeta(u), x]$, this is maximized at $x_1 = x$. As $x \geq \zeta(u)$, Lemma 4.12 ensures that $\tilde{\rho}(x) \leq cx$ for some constant $c = c(a, \sigma)$ independent of $\zeta(0)$. Proposition 4.7 and the fact that $t - u \geq t/2$ then show

$$P^0(A_u \mid \sigma(B_u)) \leq c' \frac{x}{\sqrt{t}}, \quad \text{on } \{B_u \leq x\} \quad (\text{A.60})$$

for some constant $c' = c'(a, \sigma)$. Plugging this in the expectation above, the claim follows by noting that $P^0(-\zeta(0) < B_u \leq x) \leq 2x/\sqrt{u}$. \square

With Lemma A.11 in hand, we are now ready to tackle to proof of the main claim:

Proof of Proposition 4.13. Let us augment our earlier notation by writing

$$A_u^\pm := \left\{ \min_{u \leq s \leq t} [B_s \pm \zeta(s)] > 0 \right\}. \quad (\text{A.61})$$

Our goal is to bound the probability $P^0(A_0^+ \setminus A_u^-)$. First we note that, by (A.3),

$$P^0(A_0^+) \geq \frac{1}{3} \frac{\zeta(0)}{\sqrt{t}} \quad \text{whenever } t > \zeta(0)^2. \quad (\text{A.62})$$

The condition $t > \zeta(0)^2$ will be ensured by assuming $c' > \frac{1}{2}\zeta(0)^2$. Lemma A.11 then shows

$$\frac{P^0(A_0^+ \cap \{B_u \leq x\})}{P^0(A_0^+)} \leq 3c_1 \frac{1}{\zeta(0)} \frac{x^2}{\sqrt{u}} \quad (\text{A.63})$$

whenever $u \in [c', t/2]$ and $x \geq \zeta(u) \vee 1$. To get the claim, it thus suffices to derive a good lower bound on the conditional probability $P(A_u^- \cap \{B_u \geq x\} | A_0^+)$.

For simplicity of certain bounds later, we may and will assume that $x > \zeta(u) \vee e$. For any $0 \leq u \leq t$ denote $A_{0,u}^+ := \{B_s > -\zeta(s) : 0 \leq s \leq u\}$. Then $A_0^+ \cap A_u^- = A_{0,u}^+ \cap A_u^-$ and so setting $\mathcal{F}_u := \sigma(B_s : 0 \leq s \leq u)$ and noting that $A_{0,u}^+ \in \mathcal{F}_u$,

$$P^0(A_0^+ \cap A_u^- \cap \{B_u \geq x\}) = E^0 \left(1_{A_{0,u}^+ \cap \{B_u \geq x\}} P^0(A_u^- | \mathcal{F}_u) \right). \quad (\text{A.64})$$

We will now derive a uniform estimate on the conditional probability on the right. First we note that, on $\{B_u = x_1\}$ we have

$$P^0(A_u^- | \mathcal{F}_u) = P^{x_1}(B_s \geq \zeta_u(s) : s \in [0, t-u]) \quad (\text{A.65})$$

where, as before, $\zeta_u(s) := \zeta(u+s)$. Thanks to Proposition 4.7 and Lemma 4.12 we then get, for some constant $c_2 = c_2(a, \sigma) \in (0, \infty)$,

$$P^0(A_u^- | \mathcal{F}_u) \geq \sqrt{\frac{2}{\pi}} \frac{B_u}{\sqrt{t}} \left(1 - \frac{x^2}{t} - c_2 \left(\frac{\zeta(u) + \log x}{x} \right)^{2/3} \right) \quad \text{on } \{B_u \geq x\}. \quad (\text{A.66})$$

Similarly, Proposition 4.9 and Lemma 4.12 yield

$$P^0(A_u^+ | \mathcal{F}_u) \leq \sqrt{\frac{2}{\pi}} \frac{B_u}{\sqrt{t}} \left(1 + c_3 \left(\frac{\zeta(u) + \log x}{x} \right)^{2/3} \right) \quad \text{on } \{B_u \geq x\}, \quad (\text{A.67})$$

for some constant $c_3 = c_3(a, \sigma) \in (0, \infty)$. It then follows that

$$P^0(A_0^+ \cap A_u^- \cap \{B_u \geq x\}) \geq \frac{1 - \frac{x^2}{t} - c_2 \left(\frac{\zeta(u) + \log x}{x} \right)^{2/3}}{1 + c_3 \left(\frac{\zeta(u) + \log x}{x} \right)^{2/3}} P^0(A_0^+ \cap \{B_u \geq x\}). \quad (\text{A.68})$$

In combination with (A.63), we then get

$$\frac{P^0(A_0^+ \setminus A_u^-)}{P^0(A_0^+)} \leq 3c_1 \frac{1}{\zeta(0)} \frac{x^2}{\sqrt{u}} + \frac{x^2}{t} + (c_2 + c_3) \left(\frac{\zeta(u) + \log x}{x} \right)^{2/3}. \quad (\text{A.69})$$

subject to $u \in [c', t/2]$ and $x \geq \zeta(u) \vee 1$.

Proposition 4.9 gives $P^0(A_0^+) \leq c_4/\sqrt{t}$ for some $c_4 = c_4(a, \sigma, \zeta(0))$ and so $P^0(A_0^+ \setminus A_u^-)$ is bounded by c_4/\sqrt{t} times the expression on the right of (A.69). We then choose $x := u^{\frac{7}{32}}$ and note that the first term then dominates the other two as soon as u is sufficiently large. The claim then follows by noting that $x^2/\sqrt{u} = u^{-\frac{1}{16}}$ for our choice of x . \square

Having dealt with entropic repulsion of unconditioned paths, the claim for the Brownian bridge follows readily as well:

Proof of Proposition 4.14. Consider the events

$$A_1 := \left\{ \min_{s \leq t/2} [B_s + \zeta(s)] > 0 > \min_{u \leq s \leq t/2} [B_s - \zeta(s)] \right\} \quad (\text{A.70})$$

and

$$A_2 := \left\{ \min_{s \leq \frac{1}{4}t} [B_s + \zeta(s)] > 0 \right\}. \quad (\text{A.71})$$

The event in (4.35) is then contained in $(A_1 \cap A'_2) \cup (A_2 \cap A'_1)$ and so, by the union bound and the decoupling trick in Lemma A.5, its probability is bounded by $2\sqrt{8}P^0(A_1)P^0(A_2)$. Then $P^0(A_1)$ is bounded using Proposition 4.13 while $P^0(A_2)$ using Proposition 4.9. \square

APPENDIX B: USEFUL PROPERTIES AND BOUNDS

In this short section we collect various useful facts relevant for the study of the DGFF. We also restate the results about the behavior of its extreme values that are used in this work. Having these explicated here will ease referencing throughout the rest of the article.

B.1 Gaussian processes.

We begin with two standard results concerning boundedness and continuity of rather general Gaussian processes. A good reference for this material are the books of Adler [2] and the introductory part of Adler and Taylor [3].

Lemma B.1 (Fernique majorization) *There is $K \in (0, \infty)$ such that the following holds: Let $X = \{X_t : t \in \mathfrak{X}\}$ be a separable centered Gaussian field indexed by points in a totally-bounded (pseudo)metric space (\mathfrak{X}, ρ) , where $\rho(t, s) := [E((X_t - X_s)^2)]^{1/2}$. Then for any Borel probability measure μ on \mathfrak{X}*

$$E\left(\sup_{t \in \mathfrak{X}} X_t\right) \leq K \sup_{t \in \mathfrak{X}} \int_0^\infty dr \sqrt{\log \frac{1}{\mu(B_\rho(t, r))}}, \quad (\text{B.1})$$

where $B_\rho(t, r) := \{s \in \mathfrak{X} : \rho(t, s) \leq r\}$. In addition, we also get

$$E\left(\sup_{\substack{s, t \in \mathfrak{X} \\ \rho(s, t) \leq \varepsilon}} |X_t - X_s|\right) \leq K \sup_{t \in \mathfrak{X}} \int_0^\varepsilon dr \sqrt{\log \frac{1}{\mu(B_\rho(t, r))}}. \quad (\text{B.2})$$

Proof. For (B.1) see, e.g., Adler [2, Theorem 4.1]. For (B.2) see the calculation in the proof of Adler [2, Theorem 4.5]. \square

Lemma B.2 (Borell-Tsirelson inequality) *Let \mathfrak{X} be a metric space and suppose $\{X_t : t \in \mathfrak{X}\}$ is a separable centered Gaussian process with $\sup_{t \in \mathfrak{X}} X_t < \infty$ a.s. Then*

$$P\left(\sup_{t \in \mathfrak{X}} X_t - E\left(\sup_{t \in \mathfrak{X}} X_t\right) > \lambda\right) \leq e^{-\frac{\lambda^2}{2\sigma^2}}, \quad \lambda > 0, \quad (\text{B.3})$$

where $\sigma^2 := \sup_{t \in \mathfrak{X}} E(X_t^2)$.

Proof. See, e.g., Adler [2, Theorem 2.1]. \square

B.2 Harmonic analysis.

An attractive feature of the DGFF as defined above is its connection with discrete harmonic analysis on \mathbb{Z}^2 . This is a subject that has been heavily studied in the past; see, e.g., the books by Lawler [31] and Lawler and Limić [32]. We need three objects:

- (1) the (discrete) Green function $G^D(x, y)$, defined for each $D \subsetneq \mathbb{Z}^2$, as the expected number of visits to y of the simple random walk started from x and killed upon exit from D ,
- (2) the harmonic measure $H^D(x, y)$, defined as the probability that the random walk started from x first hits $\mathbb{Z}^2 \setminus D$ at vertex y , and
- (3) the potential $\mathfrak{a}: \mathbb{Z}^2 \rightarrow [0, \infty)$ defined, e.g., by (2.8).

The simple random walk on the square lattice is recurrent and so $H^D(x, \cdot)$ is a probability measure on ∂D for each $D \subsetneq \mathbb{Z}^2$. Alternative definitions of the potential exist, e.g., using the Green function

$$\mathfrak{a}(x) = \lim_{N \rightarrow \infty} [G^{\tilde{V}_N}(0, x) - G^{\tilde{V}_N}(0, 0)], \quad (\text{B.4})$$

where $\tilde{V}_N := (-N, N)^2 \cap \mathbb{Z}^2$. The connection works the other way round as well:

Lemma B.3 *For each $D \subset \mathbb{Z}^2$ finite, the Green function G^D in D obeys*

$$G^D(x, y) = -\mathfrak{a}(x - y) + \sum_{z \in \partial D} H^D(x, z) \mathfrak{a}(y - z). \quad (\text{B.5})$$

Proof (sketch). The key is to check that $y \mapsto G^D(x, y) + \mathfrak{a}(x - y)$ is discrete harmonic on D . The stated identity then follows from the solution of a discrete Dirichlet problem. \square

The Green function $G^{\mathbb{Z}^2 \setminus \{0\}}$ is given by (2.7) which appears to have an additional term compared to (B.5). This term arises from the limit argument that is needed to make the Dirichlet problem uniquely solvable. The bound (B.5) is useful in estimates, particularly, in light of:

Lemma B.4 *The potential \mathfrak{a} admits the following asymptotic expression*

$$\mathfrak{a}(x) = g \log |x| + c_0 + O(|x|^{-2}), \quad |x| \rightarrow \infty, \quad (\text{B.6})$$

where $g = 2/\pi$ and c_0 is a constant.

Proof. This was apparently first proved by Stöhr [35] using Fourier analysis. See also Fukai and Uchiyama [27] and Kozma and Schreiber [30] for a more general approach to this. \square

Concerning the harmonic measure $H(x, \cdot)$, we need to control regularity in x uniformly in the second argument. Fortunately, it suffices to do this for square-like domains:

Lemma B.5 *Recall that $V_N := (0, N)^2 \cap \mathbb{Z}^2$ and let $a \in (0, 1/2)$. There is a constant $c = c(a)$ such that for any $x, y \in V_N$ such that $\text{dist}(x, V_N^c) \geq aN$ and $\text{dist}(y, V_N^c) \geq aN$,*

$$\max_{z \in \partial V_N} H^{V_N}(x, z) \leq \frac{c}{L} \quad (\text{B.7})$$

and

$$\max_{z \in \partial V_N} |H^{V_N}(x, z) - H^{V_N}(y, z)| \leq c \frac{|x - y|}{L^2}. \quad (\text{B.8})$$

Proof (idea). This can be proved, e.g., by invoking the continuum approximation of the harmonic measure (cf Lawler and Limić [32, Proposition 8.1.4]) and the corresponding (standard) estimate for the continuum Poisson kernel. \square

To keep out notations light, we will abuse it by occasionally writing G^D to denote also the continuum Green function. This object may as well be defined by an analogue of (B.5),

$$G^D(x, y) = -g \log |x - y| + g \int_{\partial D} \Pi^D(x, dz) \log |y - z|, \quad (\text{B.9})$$

where $\Pi^D(x, \cdot)$ is now the harmonic measure (a.k.a. Poisson kernel) associated with the standard Brownian motion killed upon exit from D . In light of (B.6), given a sequence $\{D_N\}$ of scaled-up lattice domains approximating $D \in \mathfrak{D}$ via (2.1–2.2), (B.5) converges (pointwise and locally uniformly) to (B.9) away from the diagonal in $D \times D$.

B.3 Discrete Gaussian Free Field.

We now move to the properties of the DGFF. Recall that h^D denotes the DGFF in $D \subsetneq \mathbb{Z}^2$ where we regard h^D as zero outside D . A very useful property is the behavior of h^D under conditioning on values in a subset of D , sometimes called the *domain Markov property* in the literature:

Lemma B.6 (Gibbs-Markov property) *Let $\tilde{D} \subsetneq D \subsetneq \mathbb{Z}^2$ and denote*

$$\varphi^{D, \tilde{D}}(x) := E(h^D(x) \mid \sigma(h^D(z) : z \in D \setminus \tilde{D})). \quad (\text{B.10})$$

Then we have:

- (1) *A.e. sample of $x \mapsto \varphi^{D, \tilde{D}}$ is discrete harmonic on \tilde{D} with “boundary values” determined by $\varphi^{D, \tilde{D}}(x) = h^D(x)$ for each $x \in D \setminus \tilde{D}$.*
- (2) *The field $h^D - \varphi^{D, \tilde{D}}$ is independent of $\varphi^{D, \tilde{D}}$ and, in fact, $h^D - \varphi^{D, \tilde{D}} \stackrel{\text{law}}{=} h^{\tilde{D}}$.*

Proof (idea). This is a consequence of an explicit representation of the probability law of h^D as a Gibbs measure with nearest-neighbor interactions only. \square

As a simple consequence of the Gibbs-Markov property we get:

Lemma B.7 *If $A \subseteq \tilde{D} \subseteq D \subsetneq \mathbb{Z}^2$ then*

$$P(\max_{x \in A} h^{\tilde{D}}(x) \geq \lambda) \leq 2P(\max_{x \in A} h^D(x) \geq \lambda) \quad (\text{B.11})$$

holds for each $\lambda \geq 0$.

Proof. Just write $h^D = h^{\tilde{D}} + \varphi^{D, \tilde{D}}$ and impose $\varphi^{D, \tilde{D}} \geq 0$ at the maximizer of $h^{\tilde{D}}$ on A . \square

Another useful feature of the DGFF are positive correlations. Recall that a probability measure μ on the product space $\mathbb{R}^{\mathbb{Z}^2}$ is *strong FKG* if for any finite $\Lambda \subset \mathbb{Z}^2$ and any increasing events A and B — with “increasing” defined with respect to the usual partial order on $\mathbb{R}^{\mathbb{Z}^2}$ — we have

$$\mu(A \cap B \mid \mathcal{F}_\Lambda) \geq \mu(A \mid \mathcal{F}_\Lambda) \mu(B \mid \mathcal{F}_\Lambda) \quad (\text{B.12})$$

where \mathcal{F}_Λ is the σ -algebra generated by the values of the field in Λ . We have:

Lemma B.8 (Positive correlations) *For any $D \subsetneq \mathbb{Z}^2$, the law of h^D is strong FKG.*

Proof (idea). For D finite, the probability density of h^D with respect to the product Lebesgue measure satisfies the so called FKG lattice condition which is sufficient to imply the strong FKG property. The case of infinite D is obtained by suitable limits. \square

B.4 Extreme values.

Our final set of results to be reviewed here concern the extreme values of the DGFF. Recall the notation \mathfrak{D} from Section 2.1 for the class of admissible continuum domains and $\Gamma_N^D(t)$ from (5.57) for the set of values where the DGFF in D_N is above $m_N - t$. We then have:

Lemma B.9 *For all $D \in \mathfrak{D}$ and all $t \in \mathbb{R}$,*

$$\lim_{a \rightarrow \infty} \limsup_{N \rightarrow \infty} P(|\Gamma_N^D(t)| \geq a) = 0. \quad (\text{B.13})$$

Proof. For square-like domains, this follows from Ding and Zeitouni [26, Theorem 1.2]. The extension to more general domains can be deduced from Lemma B.10 below. \square

Some applications require knowledge of the size of the intersection of $\Gamma_N^D(t)$ with an underlying set. The following comparison lemma is then quite useful:

Lemma B.10 *For each $U \subset V \subset W$, each $a \in \mathbb{R}$ and each integer $b > 0$,*

$$P(|\{x \in U : h^V(x) \geq a\}| \geq 2b) \leq 2P(|\{x \in U : h^W(x) \geq a\}| \geq b). \quad (\text{B.14})$$

Proof. This is a restatement of the last part of Lemma 3.4 in Biskup and Louidor [11]. \square

Lemma B.11 *For all $D \in \mathfrak{D}$ and all $t \in \mathbb{R}$,*

$$\lim_{r \rightarrow \infty} \limsup_{N \rightarrow \infty} P(\exists x, y \in \Gamma_N^D(c) : r < |x - y| < N/r) = 0. \quad (\text{B.15})$$

Proof. For square domains, this follows from Theorem 1.1 of Ding and Zeitouni [26] (where one allows even for t to increase as a constant times $\log \log r$). The (simple) extension to general domains $D \in \mathfrak{D}$ is provided in Proposition 3.1 of Biskup and Louidor [11]. \square

Lemma B.12 *Recall that $\alpha := 2/\sqrt{g}$ and $V_N := (0, N)^2 \cap \mathbb{Z}^2$. There is a constant $c \in (0, \infty)$ such that for all $s \geq 0$, all $t \geq 1$, all $N \geq 1$ and all sets $A \subseteq D \subseteq V_N$,*

$$P\left(\max_{x \in A} h^D(x) \geq m_N + t - s\right) \leq c \left(\frac{|A|}{N^2}\right)^{1/2} t e^{-\alpha(t-s)}. \quad (\text{B.16})$$

Proof. For $D := V_N$ this is Lemma 3.8 of Bramson, Ding and Zeitouni [15]. The more general case is implied by the bound in Lemma B.7. \square

Lemma B.13 *There are constants $c_1, c_2 \in (0, \infty)$ such that for $V_N := (0, N)^2 \cap \mathbb{Z}^2$,*

$$P\left(\left|\max_{x \in V_N} h^{V_N}(x) - m_N\right| > \lambda\right) \leq c_1 e^{-c_2 \lambda}. \quad (\text{B.17})$$

Proof. This is a restatement of Theorem 1.1 in Ding [25]. \square

Finally, let us address the passage to continuum limit. Given $\tilde{D}, D \in \mathfrak{D}$ with $\tilde{D} \subseteq D$, we will write $\varphi_N^{D, \tilde{D}}$ as a shorthand for $\varphi^{D_N, \tilde{D}_N}$. Then we have:

Lemma B.14 *Let $\tilde{D}, D \in \mathfrak{D}$ obey $\tilde{D} \subseteq D$. Then for all $x, y \in \tilde{D}$*

$$\text{Cov}(\varphi_N^{D, \tilde{D}}(\lfloor xN \rfloor), \varphi_N^{D, \tilde{D}}(\lfloor yN \rfloor)) \xrightarrow{N \rightarrow \infty} C^{D, \tilde{D}}(x, y), \quad (\text{B.18})$$

with the convergence uniform over closed subsets of $\tilde{D} \times \tilde{D}$. In particular, for $\delta > 0$ and each $N \geq 1$ there is a coupling of $\varphi_N^{D, \tilde{D}}$ and $\Phi^{D, \tilde{D}}$ such that

$$\sup_{\substack{x \in \tilde{D} \\ \text{dist}(x, \partial \tilde{D}) > \delta}} |\Phi^{D, \tilde{D}}(x) - \varphi_N^{D, \tilde{D}}(x/N)| \xrightarrow{N \rightarrow \infty} 0, \quad \text{in probability.} \quad (\text{B.19})$$

Proof (sketch). The convergence of covariances follows from the stated convergence of (B.5) to (B.9). Fix $\delta > 0$ and recall $\tilde{D}^\delta := \{x \in \mathbb{R}^2 : \text{dist}(x, \partial \tilde{D}) > \delta\}$. Given $r > 0$ small and let x_1, \dots, x_k be an r -net in \tilde{D}^δ . As convergence of the covariances implies convergence in law, and convergence in law on \mathbb{R}^n can be realized as convergence in probability, for each $N \geq 1$ there is a coupling of $\varphi_N^{D, \tilde{D}}$ and $\Phi^{D, \tilde{D}}$ such that

$$P\left(\max_{i=1, \dots, k} |\Phi^{D, \tilde{D}}(\lfloor Nx_i \rfloor) - \varphi_N^{D, \tilde{D}}(x_i)| > \varepsilon\right) \xrightarrow{N \rightarrow \infty} 0. \quad (\text{B.20})$$

The claim will then follow if we can show that

$$\lim_{r \downarrow 0} \limsup_{N \rightarrow \infty} P\left(\sup_{\substack{x, y \in \tilde{D}^\delta \\ |x-y| < r}} |\Phi^{D, \tilde{D}}(x) - \Phi^{D, \tilde{D}}(y)| > \varepsilon\right) = 0 \quad (\text{B.21})$$

and similarly for $\Phi^{D, \tilde{D}}(\cdot)$ replaced by $\varphi_N^{D, \tilde{D}}(\lfloor N \cdot \rfloor)$. This is checked using Lemmas B.1 and B.2 and some elementary regularity of $C^{D, \tilde{D}}$. \square

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